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# **SUPERSONIC FLOW OF CHEMICALLY REACTING GAS-PARTICLE MIXTURES**

**Volume I**

## **A Theoretical Analysis and Development of the Numerical Solution**

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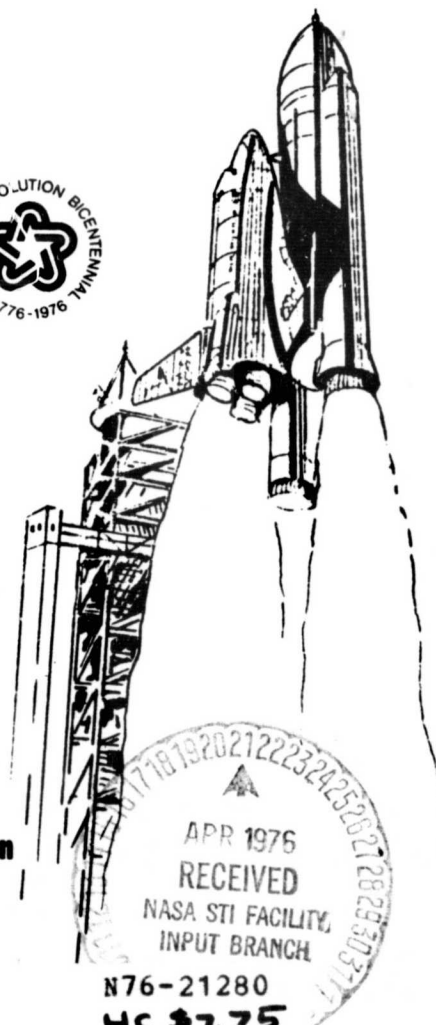
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## FOREWORD

This document is Volume I of a two-volume report describing the Reacting Multi-Phase (RAMP) Computer Code developed by the Advanced Technology Section of Lockheed's Huntsville Research & Engineering Center. Volume I addresses the theory and numerical solution for the computer code. Volume II describes the computer code along with the program input and output.

Documentation of the computer code was prepared in partial fulfillment of contract requirements (Contract NAS9-14517) with the NASA-Johnson Space Flight Center, Houston, Texas, in support of Space Shuttle related exhaust plume applications. The contracting officer's technical representative for this study was Mr. Barney B. Roberts of the Aerodynamics System Analysis Section.

The authors acknowledge the efforts of a number of individuals who contributed to the development of the RAMP code. These include Dr. Terry F. Greenwood and Mr. David C. Seymour of the NASA-Marshall Space Flight Center; and Messrs. Robert J. Prozan, Jon A. Freeman, L.R. Baker and A.W. Ratliff of Lockheed-Huntsville. Ideas and suggestions for improvement of the analysis are reflected by frequent consultation with these individuals.

Companion documents to this report include a user's manual for the RAMP code; a report which describes the modifications made to the NASA-Lewis TRAN72 computer code; and documentation of a one-dimensional solution which provides a supersonic startline for the RAMP code. These reports are, respectively:

- "Supersonic Flow of Chemically Reacting Gas-Particle Mixtures - Volume II RAMP - A Computer Code for Analysis of Chemically Reacting Gas-Particle Flows," LMSC-HREC TR D496555-II.
- "User's Guide for TRAN72 Computer Code Modified for Use with RAMP and VOFMOC Flowfield Codes," LMSC-HREC TM D390409.
- "General One-Dimensional Flow of a Gas-Particle System," LMSC-HREC TM D390876.



## SUMMARY

This report describes a numerical solution for chemically reacting supersonic gas-particle flows in rocket nozzles and exhaust plumes. The gas-particle flow solution is fully coupled in that the effects of particle drag and heat transfer between the gas and particle phases is treated. Gas and particulates exchange momentum via the drag exerted on the gas by the particles. Energy is exchanged between the phases via heat transfer (convection and/or radiation between the gas-particle phases).

Basic assumptions made in the development of the governing equations are similar to those employed by previous investigators. The primary exception is the treatment of chemical effects in the gas phase. Thermochemistry calculations (chemical equilibrium, frozen or chemical kinetics) are shown to be uncoupled from the flow solution and, as such, can be solved separately.

The solution to the set of governing equations is obtained by utilizing the method of characteristics. The equations cast in characteristic form are shown to be formally the same for ideal, frozen, chemical equilibrium and chemical non-equilibrium reacting gas mixtures. The characteristic directions for the gas-particle system are found to be the conventional gas Mach lines, the gas streamlines and the particle streamlines.

The basic mesh construction for the flow solution is along streamlines and normals to the streamlines for axisymmetric or two-dimensional flow. The analysis gives detailed information of the supersonic flow and provides for a continuous solution of the nozzle and exhaust plume flow fields. Boundary conditions for the flow solution are either the nozzle wall or the exhaust plume boundary.

The particle distribution is represented in the numerical solution by a finite distribution of particle sizes. The particle limiting streamline concept is utilized to define the region of influence for a particular particle. Particle physical and thermodynamic properties are defined by the particle mass density, distribution of particle sizes and thermodynamic data.

Presented is the development of the set of governing partial differential equations for the gas-particle system. The governing equations are cast in characteristic form and the corresponding difference equations formulated. The numerical solutions for the various point types are described and the corresponding steps of each solution outlined.

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## SYMBOLS AND NOTATION

<u>Symbol</u>	<u>Description</u>
A	parameter defined by Eq. (2.46), 1/sec; area, $\text{ft}^2$
a	speed of sound, ft/sec
$B^j$	parameter defined by Eq. (2.86c)
C	parameter defined by Eq. (2.83), $\text{ft}^2/\text{sec}^2/^\circ\text{R}$
$C_D$	drag coefficient, dimensionless
$c_p$	specific heat at constant pressure, $\text{ft}^2/\text{sec}^2/^\circ\text{R}$
$c_v$	specific heat at constant volume, $\text{ft}^2/\text{sec}^2/^\circ\text{R}$
$D_f$	drag force, lbf
$D/Dt$	substantial derivative
$dv$	an element of volume, $\text{ft}^3$
$\hat{e}$	unit vector, dimensionless
E	energy in the gas-particle system control volume, $\text{ft}^2/\text{sec}^2$
F	interpolation factor, dimensionless; or total force acting on the system defined by Eq. (2.30), lbf
f	drag coefficient parameter ( $C_D/C_{D_{\text{Stokes}}}$ ), dimensionless
G	Nusselt number parameter defined by Eq. (2.77), dimensionless
H	total enthalpy, $\text{ft}^2/\text{sec}^2$
h	local enthalpy, $\text{ft}^2/\text{sec}^2$
$I_{sp}$	specific impulse, lbf-sec/lbm
K	particle heat transfer film coefficient defined by Eq. (2.76), $\text{lbm}/\text{sec}/^\circ\text{R}/\text{ft}$
k	thermal conductivity of gas, $\text{lbm}/\text{sec}/^\circ\text{R}$

<u>Symbol</u>	<u>Description</u>
M	Mach number, dimensionless; or momentum, lbm-ft/sec
$\dot{m}$	mass flow rate, slug/sec
$m^j$	density of a particle based on a unit volume of particles, slug/ft <sup>3</sup>
NG	index denoting number of discrete gaseous species considered
NP	index denoting number of discrete particles considered
NS	index denoting number of discrete species considered = NG+NP
Nu	Nusselt number, dimensionless
$\vec{n}$	unit normal vector
P	pressure, lbf/ft <sup>2</sup>
$P_i$	partial pressure of species i, lbf/ft <sup>2</sup>
Pr	Prandtl number, dimensionless
Q	heat transferred to or from a gas-particle system control volume, Btu
q	velocity, ft/sec
R	gas constant, ft <sup>2</sup> /sec <sup>2</sup> /°R
Re	Reynolds number, dimensionless
$r^j$	particle radius, ft
S	surface area of a gas-particle system control volume; ft <sup>2</sup> ; entropy, ft <sup>2</sup> /sec <sup>2</sup> /°R
T	temperature, °R
t	time, sec
u, v	velocity components, ft/sec
V	volume of a gas-particle system control volume, ft <sup>3</sup>
W	work performed on or by the gas-particle system control volume, ft <sup>2</sup> /sec <sup>2</sup>

<u>Symbol</u>	<u>Description</u>
$\dot{w}$	production rate, lbm/sec
$x, r$	position coordinates, ft
<u>Greek</u>	
$\alpha$	accommodation coefficient, dimensionless; or Mach angle, radians
$\beta$	characteristic slope, radians
$\gamma$	specific heat ratio, dimensionless
$\Phi$	any extensive quantity of the element $dv$
$\overset{=}{\delta}$	Kronecker delta
$\delta$	0, 1 for two-dimensional or axisymmetric flow, respectively
$\overset{=}{\tau}$	viscous stress tensor, lbf/ft <sup>2</sup>
$\epsilon$	emissivity, dimensionless
$\nabla$	nabla
$\theta$	inclination of the flow vector with respect to the x-axis, radians
$\mu$	chemical potential, cal/gm
$\nu$	viscosity, lbf-sec/ft <sup>2</sup>
$\lambda$	equation modifier
$\rho$	density, slug/ft <sup>3</sup>
$\overset{=}{\sigma}$	surface stress tensor, lbf/ft <sup>2</sup>
$\sigma$	Stefan-Boltzman constant, ft <sup>2</sup> /sec <sup>3</sup>
$\psi$	particle stream function, lbm-sec/ft
$\Sigma$	indicates summation, $\sum_{j=1}^N$
$\chi$	species mass fraction
$\partial$	indicates partial derivative



### Subscripts

### Description

I	indicates initial data surface
i	indicates the quantity pertaining to species i
L	denotes local surface conditions
m	indicates the quantity pertaining to the gas-particle mixture
n	indicates n <sup>th</sup> data surface
w	denotes nozzle wall conditions
x, y	denotes partial differentiation in the x and y (or r) directions

### Superscripts

→	denotes a vector quantity
—	denotes an average value over a step length
j	indicates the quantity pertaining to a particle specie

## Section 1 INTRODUCTION

Solid propellant motors are frequently utilized to provide launch vehicle boost and stage separation. Most solid rocket motor propellants contain metal additives which increase the energy content of the system and suppress combustion pressure instabilities. The presence of metal additives results in condensed products in the nozzle-exhaust flow fields. This can create a number of adverse effects.

Condensed products (i.e., particulate aluminum oxide, etc.) being inert can do no expansion work. Consequently the particles are accelerated through the flow field via drag exerted on the gas by the viscous shearing action between the gas and particulate phases. Since the particles do no expansion work, the gas phase cools more rapidly than the particles. The particles at any given location are thus at a higher temperature than the gas so that heat is thus transferred from the particles to the gas by conduction and radiation. The net result is that the gas phase expends useful work in accelerating the particles while acquiring heat from the particles. This is an irreversible non-adiabatic process in the gas phase expansion.

Exhaust plumes create hostile environments to surfaces immersed in the flow. These are in the form of structural loads, contamination and heating. Heating results from both gas and condensates impinging on the surfaces. Particulates in the flow can also cause surface erosion and other structural damage.

Exhaust plume applications and plume impingement problems typically occur during launch, staging and rendezvous. The current Space Shuttle design offers several illustrations of areas where exhaust plume flowfield properties and flow structure must be known. This is schematically illustrated

in Fig. 1-1 which indicates potential plume related problems from solid rocket motors. During the Space Shuttle launch the solid rocket booster motor exhaust plumes impinge on the launch stand hardware. While the launch vehicle is in the vicinity of the launch pad, reradiation from the launch pad to the orbiter is a potential problem. The solid rocket motor exhaust plume affects the shuttle base drag and aerodynamics during the boost phase. The solid rocket separation motors mounted on the booster motor's forward and aft ends used to effect stage separation can potentially subject the orbiter and external hydrogen/oxygen tank to a rather severe environment.

Exhaust impingement applications require a detailed definition of the gaseous plume structure as well as particle trajectories and dynamic properties. Plume simulation studies require that plume shapes be known. The exit plane pressure must be known along with the gas thermodynamics. To adequately describe the flowfield properties, the mutual effect of gas on particle and particle on gas must be calculated. Also "real gas" effects can be significant and should be included in the gasdynamic considerations.

Solutions for the supersonic flow of a gas-particle mixture follow one of two approaches. These are: (a) a fully coupled solution in which momentum and energy are exchanged between the gas and particle phases (Refs. 1, 2 and 3), and (b) an uncoupled solution (Ref. 4) in which particle trajectories are traced through nozzle-exhaust plume flows. The uncoupled solution considers "real gas" effects; however, it treats only gas effects on the particle and is more applicable for low aluminum content propellants. The coupled solutions are primarily performance oriented. These solutions readily permit treatment of highly aluminized propellants but are restricted to constant thermodynamic properties. The performance code developed by Kliegel (Ref. 3) was subsequently utilized to trace liquid droplets in predicting contamination to surfaces (Ref. 5).

Applications discussed previously require a knowledge of the flowfield structure. Previous studies (Refs. 6, 7 and 8) indicated the need to include the treatment of "real gas" effects in nozzle-plume calculations. The

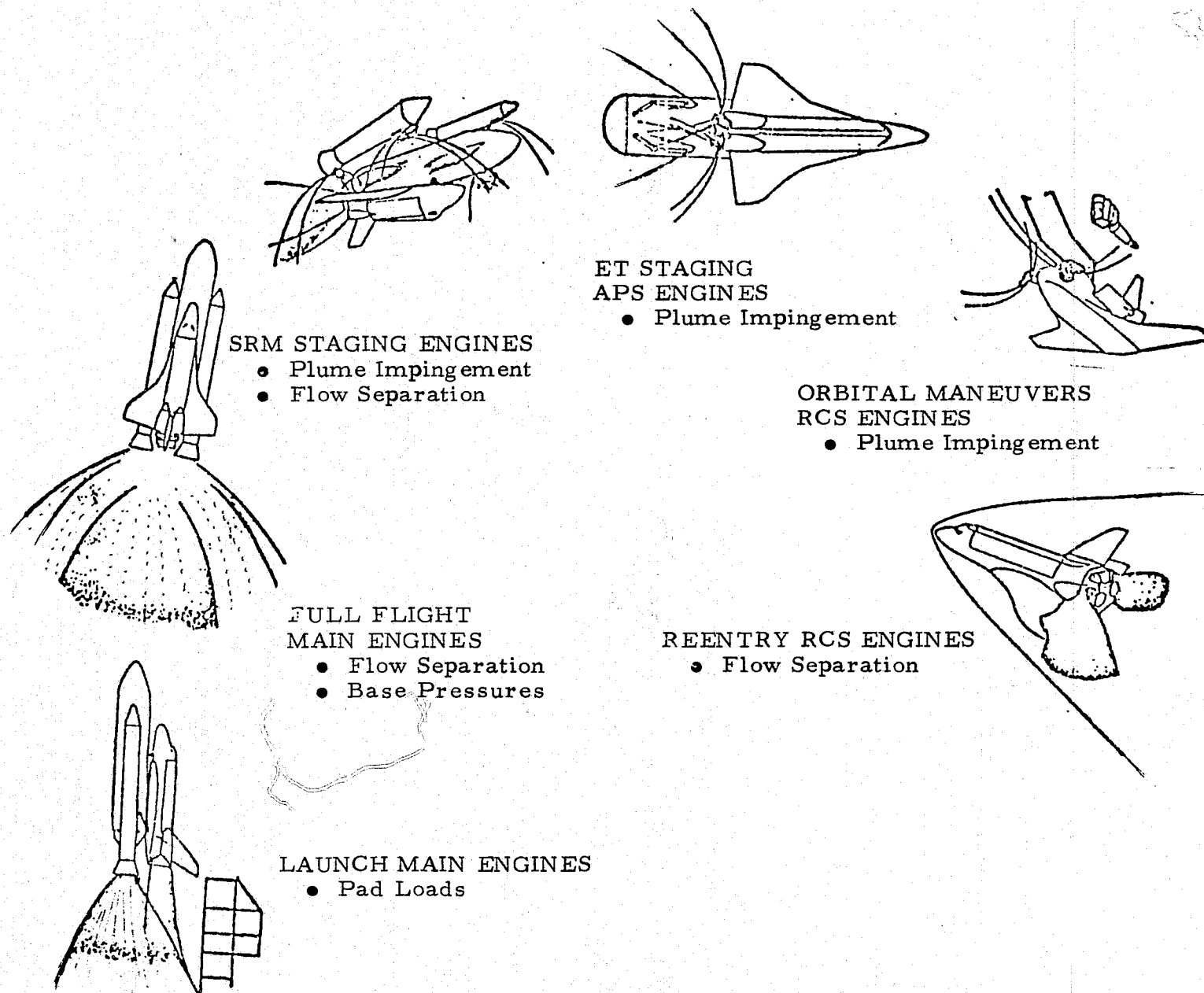


Fig. 1-1 - Mission Profile Illustrating Rocket Plumes Associated with Various Shuttle Operations

decision was subsequently made to extend the coupled gas-particle solution (Refs. 1 and 2) to include the treatment of gas thermochemistry (chemical equilibrium and chemical kinetics). The choice of computation scheme and computer code becomes more obvious after examination of anticipated applications.

The method-of-characteristics has proven to be a reliable numerical solution for nozzle-plume applications (Ref. 9). Existing gas-particle performance codes utilize the method of characteristics which indicated that an operational nozzle-exhaust plume code could be obtained with a minimum of numerical solution development. The gas-particle capability has been incorporated into a streamline-normal code (Ref. 10) which utilizes the method of characteristics to solve for local flow properties. There are several reasons for choosing this approach. These include:

- The multiple shockwave capability is treated quite readily.
- Numerical difficulties encountered in highly expanded flow are circumvented.
- The streamline-normal technique has a built-in mechanism for tracing particle streamlines.
- Transition flow between the continuum and free molecular flow regime is more readily treated.

The code in its present form will handle the flow problems which exhibit or have any of the following characteristics:

- Supersonic Inviscid Flow
- Highly Underexpanded Nozzles
- Highly Overexpanded Nozzles
- Shock Waves
- Sliplines.
- Solid Walls
- Pressure Boundary
- Nonequilibrium Chemistry (Finite Rate)

- Equilibrium Chemistry
- Ideal Chemistry
- Free Molecular Flow
- Two Phase Flow
- Oxidizer-to-Fuel (O/F) Ratio Gradients

This report presents a detailed development of the equations governing the supersonic flow of a chemically reacting gas-particle mixture. Development of the governing relations follows to a large extent the work of Kliegel. Basic assumptions are stated and the set of partial differential equations describing the gas-particle system are developed. These relations are then cast in "characteristic" form and the corresponding difference equations written. Details of the numerical solution are described for the various data point types. The presentation is then concluded with a description of the numerical integration of the conservation equations.

In addition to a description of the above analysis techniques, appendixes are included which discuss: (1) particle drag and heat transfer coefficients; (2) non-isoenergetic gas-phase flow treatment; (3) chemical equilibrium calculations in gas-particle flows; (4) non-continuum flow expansions; and (5) integration of the finite-rate chemical kinetic equations.

## Section 2

# FUNDAMENTAL EQUATIONS FOR STEADY FLOW OF REACTING GAS-PARTICLE MIXTURES

## 2.1 BASIC ASSUMPTIONS FOR THE GOVERNING EQUATIONS

The flow of a gas-particle mixture is described by the equations for conservation of mass, conservation of momentum and conservation of energy. In the gaseous phase the state variables,  $P, \rho, R$  and  $T$  are related by the equation of state while for the particulate phase the equations are for the particle drag, particle heat balance and the particle equation of state. Development of these equations is based on the following assumptions:

1. The particles are spherical in shape.
2. The particle internal temperature is uniform.
3. The gas and particles exchange thermal energy by convection and radiation (optional).
4. The gas obeys the perfect gas law and is either frozen and/or in chemical equilibrium, or is in chemical non-equilibrium.
5. The forces acting on the control volume are the pressure of the gas and the drag of the particles.
6. The gas is inviscid except for the drag it exerts on the particles.
7. There are no particle interactions.
8. The volume occupied by the particles is negligible.
9. There is no mass exchange between the phases.
10. A discrete number of particles, each of different size or chemical species, is chosen to represent the actual continuous particle distribution.
11. The particles are inert.

These are basically the same as originally stated by Kliegel except for the provision to: (a) calculate the radiation exchange between the gas and particle phase (assumption 3); (b) the gas phase can be in either frozen, in chemical

equilibrium, or chemical non-equilibrium (assumption 4). The total mass and energy of the gas-particle system are not constant since provisions are made for particle streamlines to penetrate the exhaust plume boundary.

## 2.2 GOVERNING EQUATIONS FOR THE GAS-PARTICLE MIXTURES

### 2.2.1 Continuity Equation

The various forms of the continuity equation for chemically reacting gas-particle mixtures are derived in this section. The forms of the equation derived, in order are:

1. Species Continuity Equation (2.6)
2. Global Continuity Equation/(Steady State) (2.13)/(2.27)
3. Gas Continuity Equation (2.19)
4. Particle Continuity Equation (2.20)
5. Species Continuity Equations as a Function of Mass Fraction/(Steady State) (2.25)/(2.29)
6. Particle Continuity Equation for Each Particle species  $j$ /(Steady State) (2.26)/(2.28)

Consider a chemically reacting gas-particle mixture flowing through some arbitrary stationary control volume,  $V$ , (Fig. 2-1) bounded by the control surface,  $S$ .

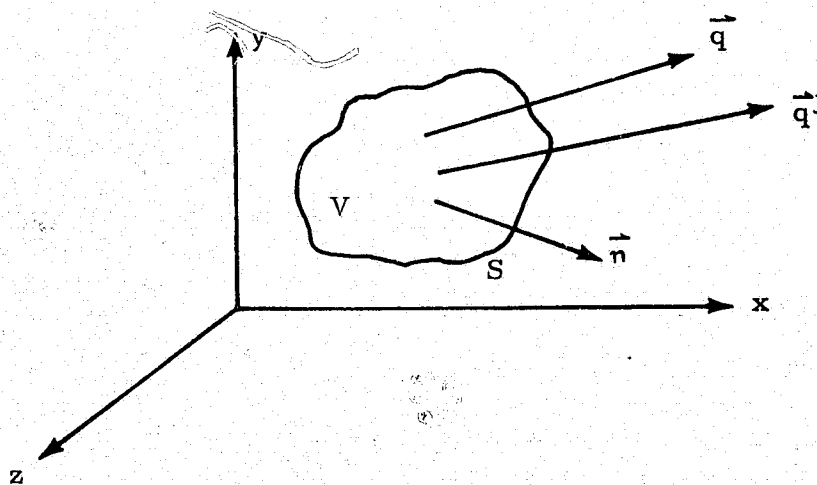


Fig. 2-1 - Control Volume for the System of Equations



For such a flow system, Reynold's Transport theorem states that:

$$\frac{D}{Dt} \int_V A dV = \int_V \frac{\partial A}{\partial t} dV + \int_S A \vec{q} \cdot \vec{n} dS; \quad (2.1)$$

where A = any arbitrary parameter.

Applying the Gauss, or Divergence theorem

$$\int_S \vec{B} \cdot \vec{n} dS = \int_V \nabla \cdot \vec{B} dV; \quad (2.2)$$

where B = any arbitrary vector quantity  
to Eq. (2.1) yields:

$$\frac{D}{Dt} \int_V A dV = \int_V \frac{\partial A}{\partial t} dV + \int_V \nabla \cdot A \vec{q} dV. \quad (2.3)$$

Furthermore, the substantial or Euler's derivative may be written in the form:

$$\frac{DA}{Dt} = \frac{\partial A}{\partial t} + \vec{q} \cdot \nabla A. \quad (2.4)$$

In a flow system of gas-particle mixtures in which chemical reactions take place, the principle of conservation of mass of each chemical species may be written as:

$$\frac{D}{Dt} \int_V \rho_i dV - \int_V \dot{w}_i dV = 0 \quad i = 1, NS; \quad (2.5)$$

where  $\dot{w}_i$  = production rate of species i due to either internal or external sources such as chemical reactions and mass additions.

Applying Eq. (2.3) to the first term of Eq. (2.5)

$$\frac{D}{Dt} \int_V \rho_i dV = \int_V \frac{\partial \rho_i}{\partial t} dV + \int_V \nabla \cdot \rho_i \vec{q}_i dV \quad i = 1, NS,$$

substituting the result back into Eq. (2.5)

$$\int_V \frac{\partial \rho_i}{\partial t} dV + \int_V \nabla \cdot \rho_i \vec{q}_i dV - \int_V \dot{w}_i dV = 0 \quad i = 1, NS,$$

and rearranging terms yields

$$\int_V \frac{\partial \rho_i}{\partial t} dV = - \int_V \nabla \cdot \rho_i \vec{q}_i dV + \int_V \dot{w}_i dV \quad i = 1, NS.$$

The above equation simply states that, for species  $i$ , the mass rate of increase inside the control volume is equal to the net rate of mass flow into the control volume plus the rate of mass produced due to chemical reactions and mass addition. In the following, we shall, however, exclude mass addition from external sources.

Combining the integrands under one integral

$$\int_V \left( \frac{\partial \rho_i}{\partial t} + \nabla \cdot \rho_i \vec{q}_i - \dot{w}_i \right) dV = 0 \quad i = 1, NS,$$

and noting that the volume  $V$  under consideration is arbitrary; the only way for the above equation to be valid for all  $V$  is for the integrand to vanish. We therefore have

$$\frac{\partial \rho_i}{\partial t} + \nabla \cdot \rho_i \vec{q}_i - \dot{w}_i = 0 \quad i = 1, NS. \quad (2.6)$$

Equation (2.6) is known as the species continuity equation. It is valid for each chemical species at each internal quantum state. We shall, however, assume that the various internal modes of motion are fully excited and are in equilibrium with each other. It is well known that this is approximately the case for the translational and rotational degrees of freedom where the equilibrium value is attained in a few collisions. In general, the vibrational degree of freedom approaches the equilibrium state somewhat more slowly,

except at very high temperatures. As chemical reactions usually occur at high temperatures, this approximation is often justified. The global continuity equation can now be derived by summing Eq. (2.6) for all species present.

$$\sum_{i=1}^{NS} \left( \frac{\partial \rho_i}{\partial t} + \nabla \cdot \rho_i \vec{q}_i - \dot{w}_i \right) = 0$$

or

$$\frac{\partial}{\partial t} \sum_{i=1}^{NS} \rho_i + \nabla \cdot \sum_{i=1}^{NS} \rho_i \vec{q}_i - \sum_{i=1}^{NS} \dot{w}_i = 0 \quad (2.7)$$

To arrive at the final form of the global continuity equation, it is necessary to take a closer look at the flow system.

In the flow system analyzed, the time variation of the thermodynamic functions is slow compared to the longest relaxation time of the system. Therefore, the assumption that thermodynamic local equilibrium exists can be made. In such a system, the thermodynamic quantities for the nonequilibrium system are the same functions of the local state variables as the corresponding equilibrium thermodynamic quantities. Therefore, the partial specific quantity  $\phi_i$  in an arbitrary volume element  $dV$  of a nonequilibrium system may be defined by the equilibrium relation

$$\bar{\phi}_i = \left( \frac{\partial \Phi}{\partial m_i} \right)_{T, P, m_j}$$

The quantities being held constant during the differentiation are the locally defined temperature  $T$ , pressure  $P$ , and the masses  $m_j$  of the other  $NS-1$  species.

The specific quantity  $\phi_m$  ( $\Phi$  per unit mass) is given in terms of the partial specific quantities  $\bar{\phi}_i$  by

$$\rho_m \phi_m = \sum_i^{NS} \rho_i \bar{\phi}_i \quad i = 1, NS;$$

where the relationship between the mass density  $\rho_m$  of the mixture and the species density  $\rho_i$  is given by

$$\rho_m = \sum_{i=1}^{NS} \rho_i = \text{fluid density} \quad (2.8)$$

The specific velocity,  $\vec{q}_m$ , specific enthalpy,  $h_m$ , and specific internal energy,  $e_m$  are given by:

$$\rho_m \vec{q}_m = \sum_{i=1}^{NS} \rho_i \vec{q}_i \quad (2.9)$$

$$\rho_m h_m = \sum_{i=1}^{NS} \rho_i h_i \quad (2.10)$$

and

$$\rho_m e_m = \sum_{i=1}^{NS} \rho_i e_i \quad (2.11)$$

Substituting Eqs. (2.8) and (2.9) into (2.7),

$$\frac{\partial \rho_m}{\partial t} + \nabla \cdot \rho_m \vec{q}_m - \sum_{i=1}^{NS} \dot{w}_i = 0 ,$$

and noting that for a closed system, the total mass for chemical reactions is conserved

$$\sum_{i=1}^{NS} \dot{w}_i = 0 ; \quad (2.12)$$

Equation (2.7) reduces to

$$\frac{\partial \rho_m}{\partial t} + \nabla \cdot \rho_m \vec{q}_m = 0 . \quad (2.13)$$

Equation (2.13) is known as the global continuity equation. It can be written alternatively as

$$\frac{D\rho_m}{Dt} + \rho_m \nabla \cdot \vec{q}_m = 0$$

by means of the substantial derivative.

Equations (2.8), (2.9) and (2.12) can be rewritten in terms of gaseous and particle species as follows:

Equation (2.8)

$$\rho_m = \sum_{i=1}^{NS} \rho_i = \rho + \sum_{j=1}^{NP} \rho^j ; \quad (2.14)$$

Equation (2.9):

$$\rho_m \vec{q}_m = \sum_{i=1}^{NS} \rho_i \vec{q}_i = \rho \vec{q} + \sum_{j=1}^{NP} \rho^j \vec{q}^j \quad (2.15)$$

Equation (2.12):

$$\sum_{i=1}^{NS} \dot{w}_i = 0 = \dot{w} + \sum_{j=1}^{NP} \dot{w}^j = 0 \quad (2.16)$$

Assuming that reactions of the form  $A(\text{gas}) + B(\text{gas}) \rightleftharpoons C(\text{particle}) + D(\text{particle})$  do not contribute significantly to the system (interchange of phases is negligible due to chemical reactions), Eq. (2.16) implies that

$$\dot{w} = 0 \quad \text{and} \quad \sum_{j=1}^{NP} \dot{w}^j = 0 . \quad (2.17)$$

Substituting Eqs. (2.14) through (2.17) into Eq. (2.7)

$$\frac{\partial}{\partial t} \left( \rho + \sum_{j=1}^{NP} \rho^j \right) + \nabla \cdot \left( \rho \vec{q} + \sum_{j=1}^{NP} \rho^j \vec{q}^j \right) - \cancel{\dot{w}} - \sum_{j=1}^{NP} \cancel{\dot{w}^j} = 0 ,$$

and rearranging terms yields

$$\frac{\partial \rho}{\partial t} + \nabla \cdot \rho \vec{q} + \sum_{j=1}^{NP} \left( \frac{\partial \rho^j}{\partial t} + \nabla \cdot \rho^j \vec{q}^j \right) = 0 \quad (2.18)$$

Since the gaseous species and particle species do not interchange phases, both contributions must vanish. Therefore

$$\frac{\partial \rho}{\partial t} + \nabla \cdot \rho \vec{q} = 0 \quad (2.19)$$

and

$$\sum_{j=1}^{NP} \left( \frac{\partial \rho^j}{\partial t} + \nabla \cdot \rho^j \vec{q}^j \right) = 0 \quad (2.20)$$

Equation (2.19) is known as the gas continuity equation and is valid when applied to the system of gas species. Equation (2.20) is known as the particle continuity equation and is valid when applied to the system of particle species.

The species continuity Eq. (2.6) may be written in a more convenient form by making use of the gas continuity equation and introducing the mass fraction of species  $i$ . In general, the mass fraction of species  $i$  may be defined as:

$$X_i = \frac{\rho_i}{\rho_m} \quad (2.21)$$

Substituting  $X_i$  into the species continuity Eq. (2.6)

$$\frac{\partial}{\partial t} (X_i \rho_m) + \nabla \cdot X_i \rho_m \vec{q}_i - \dot{w}_i = 0; \quad i = 1, NS$$

expanding the first term

$$\rho_m \frac{\partial X_i}{\partial t} + X_i \frac{\partial \rho_m}{\partial t} + \nabla \cdot X_i \rho_m \vec{q}_i - \dot{w}_i = 0 \quad i = 1, NS \quad (2.21a)$$

and applying the vector identity

$$\nabla \cdot X_i \rho_m \vec{q}_i = \nabla \cdot (X_i (\rho_m \vec{q}_i)) = \nabla X_i \cdot \rho_m \vec{q}_i + X_i (\nabla \cdot \rho_m \vec{q}_i)$$

results in

$$\rho_m \left( \frac{\partial X_i}{\partial t} + \vec{q}_i \cdot \nabla X_i \right) + X_i \left( \frac{\partial \rho_m}{\partial t} + \nabla \cdot \rho_m \vec{q}_i \right) - \dot{w}_i = 0 \quad i = 1, NS \quad (2.22)$$

By limiting the system to only gas phase reactions (no reactions in the particulate phase) the species mass fraction  $X_i$  may be redefined as:

$$X_i = \rho_i / \rho \quad (2.23)$$

Assuming all gaseous species have velocity  $\vec{q}_i = \vec{q}$ , Eq. (2.22) becomes

$$\rho \left( \frac{\partial X_i}{\partial t} + \vec{q} \cdot \nabla X_i \right) + X_i \left( \frac{\partial \rho}{\partial t} + \nabla \cdot \rho \vec{q} \right) - \dot{w}_i = 0 \quad i = 1, NS \quad (2.24)$$

Applying the gas continuity equation to the above expression the second term is set equal to zero and Eq. (2.24) becomes:

$$\rho \left( \frac{\partial X_i}{\partial t} + \vec{q} \cdot \nabla X_i \right) - \dot{w}_i = 0 \quad i = 1, NS$$

The final form of Eq. (2.24) is obtained by applying substantial derivative, Eq. (2.4) to yield

$$\rho \frac{DX_i}{Dt} - \dot{w}_i = 0 \quad i = 1, NS \quad \begin{array}{l} \text{species continuity} \\ \text{equation} \end{array} \quad (2.25)$$

Under the above assumption that there are no particle reactions, Eq. (2.20) for each particle species  $j$  becomes:

$$\frac{\partial \rho^j}{\partial t} + \nabla \cdot \rho^j \vec{q}^j = 0 \quad j = 1, NP \quad \begin{array}{l} \text{particle continuity} \\ \text{equation} \end{array} \quad (2.26)$$

For steady state flow processes, Eqs. (2.13), (2.26) and (2.25) become:

Equation (2.13)

$$\nabla \cdot \rho_m \vec{q}_m = 0 \quad \text{global continuity equation} \quad (2.27)$$

Equation (2.26)

$$\nabla \cdot \rho^j \vec{q}^j = 0 \quad j = 1, NP \quad \text{particle continuity equation, and} \quad (2.28)$$

Equation (2.25)

$$\rho \vec{q}_i \cdot \nabla X_i - \dot{w}_i = 0 \quad i = 1, NS \quad \text{species continuity equation} \quad (2.29)$$

## 2.2.2 Momentum Equation

The various forms of the global momentum equation for a chemically reacting gas-particle mixture are derived in this section. The forms of the equation derived, in order are:

1. General Global Momentum Equation (2.42)
2. General Global Momentum Equation with particle drag effects (2.49)
3. Global Momentum Equation for steady state, inviscid, no body force flow (2.50)

The linear momentum of an element of mass  $m$  is a vector quantity defined as  $m\vec{q}_m$ . The fundamental statement of Newton's law for an inertial reference is given in terms of momentum as

$$\vec{F} - \frac{D}{Dt} (m_m \vec{q}_m) = 0 \quad (2.30)$$

To derive the general global momentum equation the two terms of Eq. (2.30) will be analyzed separately.



The total force acting on the system may be represented as the sum of the surface force distributions (force distributions acting on the boundary of the system) and body force distributions (force distributions acting on the material inside the system) as follows:

$$\vec{F} = \int_S \vec{\sigma} \cdot \vec{n} dS + \int_V \sum_{i=1}^{NS} \rho_i \vec{f}_i dV \quad (2.31)$$

The surface force term may be converted into a volume integral by applying Gauss' theorem, Eq. (2.2). Therefore,

$$\vec{F} = \int_V \nabla \cdot \vec{\sigma} dV + \int_V \sum_{i=1}^{NS} \rho_i \vec{f}_i dV \quad (2.32)$$

The surface stress tensor,  $\vec{\sigma}$  may be written in the form:

$$\vec{\sigma} = -P\vec{\delta} + \vec{\tau} \quad (2.33)$$

and

$$\nabla \cdot \vec{\sigma} = -\nabla P + \nabla \cdot \vec{\tau} \quad (2.34)$$

Substituting Eq. (2.34) into Eq. (2.32), the total force acting on the system may be written in the form

$$\vec{F} = \int_V (-\nabla P + \nabla \cdot \vec{\tau}) dV + \int_V \sum_{i=1}^{NS} \rho_i \vec{f}_i dV ;$$

or, upon combining terms

$$\vec{F} = \int_V \left( -\nabla P + \nabla \cdot \vec{\tau} + \sum_{i=1}^{NS} \rho_i \vec{f}_i \right) dV \quad (2.35)$$

The momentum term  $m_m \vec{q}_m$  in Eq. (2.30) may be rewritten in the form

$$m_m \vec{q}_m = \int_V \sum_{i=1}^{NS} \rho_i \vec{q}_i dV ;$$

from which the substantial derivative can then be written

$$\frac{D}{Dt} (m_m \vec{q}_m) = \frac{D}{Dt} \int_V \sum_{i=1}^{NS} \rho_i \vec{q}_i dV . \quad (2.36)$$

Applying Reynold's transport theorem Eq. (2.3) to Eq. (2.36)

$$\frac{D}{Dt} (m_m \vec{q}_m) = \int_V \frac{\partial}{\partial t} \sum_{i=1}^{NS} \rho_i \vec{q}_i dV + \int_V \nabla \cdot \sum_{i=1}^{NS} \rho_i \vec{q}_i \vec{q}_i dV ;$$

rearranging terms

$$\frac{D}{Dt} (m_m \vec{q}_m) = \int_V \left\{ \frac{\partial}{\partial t} \sum_{i=1}^{NS} \rho_i \vec{q}_i + \nabla \cdot \sum_{i=1}^{NS} \rho_i \vec{q}_i \vec{q}_i \right\} dV ; \quad (2.37)$$

and rewriting the results in terms of the gaseous and particle species results in:

$$\frac{D}{Dt} (m_m \vec{q}_m) = \int_V \left\{ \frac{\partial}{\partial t} \left( \rho \vec{q} + \sum_{j=1}^{NP} \rho^j \vec{q}^j \right) + \nabla \cdot \left( \rho \vec{q} \vec{q} + \sum_{j=1}^{NP} \rho^j \vec{q}^j \vec{q}^j \right) \right\} dV$$

or,

$$\frac{D}{Dt} (m_m \vec{q}_m) = \int_V \left\{ \frac{\partial}{\partial t} (\rho \vec{q}) + \nabla \cdot \rho \vec{q} \vec{q} + \sum_{j=1}^{NP} \left[ \frac{\partial}{\partial t} (\rho^j \vec{q}^j) + \nabla \cdot \rho^j \vec{q}^j \vec{q}^j \right] \right\} dV \quad (2.38)$$

Substituting Eqs. (2.35) and (2.38) into Eq. (2.30), dropping the integral notation and rearranging the terms yields

$$\frac{\partial}{\partial t} (\rho \vec{q}) + \nabla \cdot \rho \vec{q} \vec{q} + \sum_{j=1}^{NP} \left[ \frac{\partial}{\partial t} (\rho^j \vec{q}^j) + \nabla \cdot \rho^j \vec{q}^j \vec{q}^j \right] = -\nabla P + \nabla \cdot \vec{\tau} + \sum_{i=1}^{NS} \rho_i \vec{f}_i \quad (2.39)$$

Expanding the first two terms of Eq. (2.39)

$$\frac{\partial}{\partial t} (\rho \vec{q}) + \nabla \cdot \rho \vec{q} \vec{q} = \rho \frac{\partial \vec{q}}{\partial t} + \vec{q} \frac{\partial \rho}{\partial t} + \rho \vec{q} \cdot \nabla \vec{q} + \vec{q} \nabla \cdot \rho \vec{q}; \quad (2.39a)$$

and then applying the gas continuity Eq. (2.19)

$$\frac{\partial \rho}{\partial t} + \nabla \cdot \rho \vec{q} = 0$$

to the results, yields

$$\frac{\partial}{\partial t} (\rho \vec{q}) + \nabla \cdot \rho \vec{q} \vec{q} = \rho \left( \frac{\partial \vec{q}}{\partial t} + \vec{q} \cdot \nabla \vec{q} \right) \quad (2.40)$$

The third term of Eq. (2.39) may be manipulated in a like manner using the particle continuity Eq. (2.26) to yield:

$$\sum_{j=1}^{NP} \left[ \frac{\partial}{\partial t} (\rho^j \vec{q}^j) + \nabla \cdot \rho^j \vec{q}^j \vec{q}^j \right] = \sum_{j=1}^{NP} \rho^j \left( \frac{\partial \vec{q}^j}{\partial t} + \vec{q}^j \cdot \nabla \vec{q}^j \right) \quad (2.41)$$

Substituting Eqs. (2.40) and (2.41) back into Eq. (2.39) results in:

$$\rho \left( \frac{\partial \vec{q}}{\partial t} + \vec{q} \cdot \nabla \vec{q} \right) + \sum_{j=1}^{NP} \rho^j \left( \frac{\partial \vec{q}^j}{\partial t} + \vec{q}^j \cdot \nabla \vec{q}^j \right) = -\nabla P + \nabla \cdot \vec{\tau} + \sum_{i=1}^{NS} \rho_i \vec{f}_i;$$

or in terms of the substantial derivative, Eq. (2.4),

$$\rho \frac{D}{Dt} \vec{q} + \sum_{j=1}^{NP} \rho^j \frac{D}{Dt} \vec{q}^j = -\nabla P + \nabla \cdot \vec{\tau} + \sum_{i=1}^{NS} \rho_i \vec{f}_i \quad (2.42)$$

Equation (2.42) is the general global momentum equation. In this equation, the five terms are, respectively: (1) the non-stationary and convective rate of change of momentum per unit volume of the gaseous species; (2) the non-stationary and convective rate of change of momentum per unit volume of the particle species; (3) the net hydrostatic pressure force; (4) the viscous stress force acting on the surface of the unit volume; and (5) the body forces per unit volume.

Since it is assumed that all forces are negligible, except that of the pressure of the gas and the particle drag, the only force exerted on the particle is a drag force  $\vec{D}_f^j$  caused by the relative motion between the gas and the particle. Figure 2-2 illustrates a spherical particle in a gas flow field.

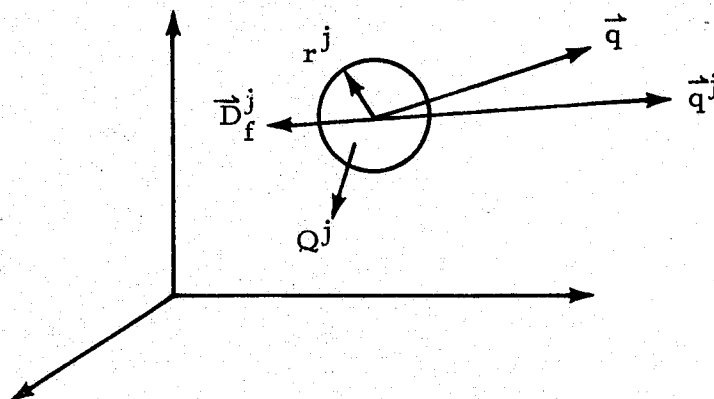


Fig. 2-2 - Particle Momentum and Heat Transfer Model

To determine the effects of particle drag on the global momentum equation, Newton's Second Law is applied to the drag forces  $\vec{D}_f^j$  acting on particle species  $j$ .

$$\vec{D}_f^j = \frac{D}{Dt} (m\vec{q})^j$$

or rewritten in terms of particle density

$$\vec{D}_f^j = \frac{D}{Dt} \left[ m^j \frac{4}{3} \pi (r^j)^3 \vec{q}^j \right] = \frac{4}{3} \pi (r^j)^3 m^j \frac{D}{Dt} \vec{q}^j.$$

In general, the drag force on a spherical particle may be written in the form:

$$\vec{D}_f^j = C_D^j \pi (r^j)^2 \left( \frac{1}{2} \rho \Delta \vec{q}^j |\Delta \vec{q}^j| \right)$$

where

$$\Delta \vec{q}^j = \vec{q} - \vec{q}^j. \quad (2.43)$$

Equating the two expressions for particle drag

$$\frac{4}{3} \pi (r^j)^3 m^j \frac{D}{Dt} \vec{q}^j = C_D^j \pi (r^j)^2 \frac{1}{2} \rho \Delta \vec{q}^j |\Delta \vec{q}^j|;$$

and rearranging terms yields:

$$\frac{D}{Dt} \vec{q}^j = \frac{3}{8} \left( \frac{\rho C_D^j}{m^j r^j} \right) \Delta \vec{q}^j |\Delta \vec{q}^j| \quad (2.44)$$

Equation (2.44) can be simplified by referencing to the Stokes flow regime (Reynolds number is less than 1) in the following manner.

Define

$$f^j = \frac{C_D^j}{C_{D_{Stokes}}}$$

where

$$C_{D_{Stokes}} = \frac{24}{Re}, \quad Nu = 2.$$

Therefore,

$$C_D^j = \frac{24 f^j}{Re} = 24 f^j \left( \frac{\nu}{2 r^j \rho |\Delta \vec{q}^j|} \right). \quad (2.45)$$

Substituting Eq. (2.45) into Eq. (2.44) and defining the parameter

$$A^j = \frac{9}{2} \left[ \frac{\nu f^j}{m^j r_j^2} \right]; \quad (2.46)$$

the momentum balance becomes

$$\frac{D}{Dt} \vec{q}^j = A^j \Delta \vec{q}^j. \quad (2.47)$$

Expanding the substantial derivative by means of Eq. (2.4), Eq. (2.47) becomes

$$\frac{\partial}{\partial t} \vec{q}^j + \vec{q}^j \cdot \nabla \vec{q}^j = A^j \Delta \vec{q}^j = \frac{D \vec{q}^j}{Dt}. \quad (2.48)$$

Substituting Eq. (2.47) into the general global momentum equation results in:

$$\rho \frac{D \vec{q}}{Dt} + \sum_{j=1}^{NP} \rho^j A^j \Delta \vec{q}^j = -\nabla P + \nabla \cdot \bar{\bar{\tau}} + \sum_{i=1}^{NS} \rho_i \vec{f}_i \quad (2.49)$$

Equation (2.49) is the general global momentum equation with particle drag effects.

For the case in which the flow may be described as steady state, inviscid and no body forces present Eq. (2.49) becomes:

$$\rho \vec{q} \cdot \nabla \vec{q} + \sum_{j=1}^{NP} \rho^j A^j \Delta \vec{q}^j = -\nabla P. \quad (2.50)$$

We note at this point that chemical reactions do not alter the forms of the global continuity equation or equations of motion.

Equation (2.50) will be expanded now for later use in the derivation of the energy equation. For an arbitrary vector  $\vec{A}$  the following identity exists

$$\vec{A} \cdot \nabla \vec{A} = \frac{1}{2} \nabla (\vec{A} \cdot \vec{A}) - \vec{A} \times (\nabla \times \vec{A}). \quad (2.51)$$

Applying this identity to the first term in Eq. (2.50)

$$\rho \vec{q} \cdot \nabla \vec{q} = \frac{1}{2} \rho \nabla (\vec{q} \cdot \vec{q}) - \rho \vec{q} \times (\nabla \times \vec{q}) ;$$

substituting the results back into Eq. (2.50)

$$\frac{1}{2} \rho \nabla (\vec{q} \cdot \vec{q}) - \rho \vec{q} \times (\nabla \times \vec{q}) + \sum_{j=1}^{NP} \rho^j A^j \Delta \vec{q}^j = - \nabla P ;$$

and rearranging terms yields

$$\frac{1}{2} \nabla (\vec{q} \cdot \vec{q}) = \vec{q} \times (\nabla \times \vec{q}) - \frac{1}{\rho} \sum_{j=1}^{NP} \rho^j A^j \Delta \vec{q}^j - \frac{\nabla P}{\rho} \quad (2.52)$$

### 2.2.3 Energy Equation

The various forms of the energy equation for a chemically reacting gas-particle mixture are derived in this section. The forms of the equation derived are:

1. General Global Energy Equation Neglecting the Effects of Radiation (2.74)
2. General Particle Energy Balance Equation (2.84)
3. General Global Energy Equation with Radiation Effects (2.85)
4. Global Energy Equation with Radiation Effects for Flow Described as Steady State, Adiabatic, Inviscid, and No Body Forces Present (2.105)

The most general form of the energy equation, according to the first law of thermodynamics or the law of conservation of energy can be written as

$$\frac{DE}{Dt} - \frac{dQ}{dt} + \frac{dW}{dt} = 0 \quad (2.53)$$

The energy  $E$  in the control volume may be written in terms of the species present as

$$E = \int_V \sum_{i=1}^{NS} \rho_i e_i dV ;$$

where

$$e_i = i_i + \frac{q_i^2}{2} + \frac{q_i}{g_c} Z_i \quad i = 1, NS \quad (2.54)$$

$i_i$  = internal energy of species  $i$  per unit mass

$\frac{q_i^2}{2}$  = kinetic energy of species  $i$  per unit mass

$\frac{q_i}{g_c} Z_i$  = potential energy of species  $i$  per unit mass.

The substantial derivative can then be written

$$\frac{DE}{Dt} = \frac{D}{Dt} \int_V \sum_{i=1}^{NS} \rho_i e_i dV ;$$

and expanded by applying Eq. (2.3) to yield

$$\frac{DE}{Dt} = \int_V \frac{\partial}{\partial t} \sum_{i=1}^{NS} \rho_i e_i dV + \int_V \nabla \cdot \sum_{i=1}^{NS} \rho_i e_i \vec{q}_i dV . \quad (2.55)$$

Recalling that the total force acting on the control volume according to Eq. (2.31) is

$$\vec{F} = \int_S \vec{\sigma} \cdot \vec{n} dS + \int_V \sum_{i=1}^{NS} \rho_i \vec{f}_i dV ;$$

and applying the definition for the rate at which work is performed on or by the control volume  $dW/dt = -\vec{q} \cdot \vec{F}$ ,



one obtains

$$\frac{dW}{dt} = - \int_S \vec{q} \cdot \vec{\sigma} \cdot \vec{n} dS - \int_V \sum_{i=1}^{NS} \vec{q}_i \cdot \rho_i \vec{f}_i dV$$

The surface integral may be converted into a volume integral by applying Gauss' theorem, Eq. (2.2). Therefore,

$$\frac{dW}{dt} = - \int_V \nabla \cdot (\vec{\sigma} \cdot \vec{q}) dV - \int_V \sum_{i=1}^{NS} \vec{q}_i \cdot \rho_i \vec{f}_i dV.$$

Recalling the definition of the surface stress tensor  $\vec{\sigma}$ , Eq. (2.33)

$$\vec{\sigma} = -P\vec{\delta} + \vec{\tau},$$

and taking the dot product

$$\vec{\sigma} \cdot \vec{q} = -P\vec{q} + \vec{\tau} \cdot \vec{q},$$

the first volume integral in the above equation may be written as

$$\int_V \nabla \cdot (-P\vec{q} + \vec{\tau} \cdot \vec{q}) dV. \quad (2.56)$$

Substituting Eq. (2.56) back into the expression for  $dW/dt$

$$\frac{dW}{dt} = - \int_V \nabla \cdot (-P\vec{q} + \vec{\tau} \cdot \vec{q}) - \int_V \sum_{i=1}^{NS} \vec{q}_i \cdot \rho_i \vec{f}_i dV,$$

combining integrals

$$\frac{dW}{dt} = - \int_V \left[ \nabla \cdot (-P\vec{q} + \vec{\tau} \cdot \vec{q}) + \sum_{i=1}^{NS} \vec{q}_i \cdot \rho_i \vec{f}_i \right] dV,$$

and expanding terms yields

$$\frac{dW}{dt} = \int_V \left[ \nabla \cdot P\vec{q} - \nabla \cdot (\vec{\tau} \cdot \vec{q}) - \sum_{i=1}^{NS} \vec{q}_i \cdot \rho_i \vec{f}_i \right] dV \quad (2.57)$$

The rate at which heat is transferred to or from the control volume may be written in the form

$$\frac{dQ}{dt} = \int_S \vec{Q} \cdot \vec{n} dS ,$$

or in terms of a volume integral by applying Gauss' theorem, Eq.(2.2)

$$\frac{dQ}{dt} = \int_V \nabla \cdot \vec{Q} dV , \quad (2.58)$$

where  $\vec{Q}$  is the conduction heat flux vector.

Substituting Eqs.(2.55), (2.57) and (2.58) into Eq.(2.53) and dropping the integral notation results in

$$\frac{\partial}{\partial t} \sum_{i=1}^{NS} \rho_i e_i + \nabla \cdot \sum_{i=1}^{NS} \rho_i e_i \vec{q}_i - \nabla \cdot \vec{Q} + \nabla \cdot P\vec{q} - \nabla \cdot (\vec{\tau} \cdot \vec{q}) - \sum_{i=1}^{NS} \vec{q}_i \cdot \rho_i \vec{f}_i = 0 . \quad (2.59)$$

Equation (2.59) is the general global energy equation neglecting the effects of radiation. This equation is based on a unit volume. The six terms are, respectively: (1) and (2) the total rate of increase of internal, kinetic and potential energy caused by local and convective changes; (3) the heat conducted to or from the control volume; (4) and (5) the work done on the fluid element due to surface forces; and (6) the work done on the element by the body forces.

In deriving Eq.(2.59), all that has been neglected is the energy transferred due to radiation, which if necessary, can be added to the equation as an extra term. Radiation effects will be determined later in this section.

The microscopic quantum effects (interchange of energy and mass) are not considered in Eq.(2.59) as it has been implied by the classical first law of thermodynamics. The general global energy equation will now be expanded and written in terms of the gaseous and particle species present.

Recall that

$$e_i = i_i + \frac{q_i^2}{2} + \frac{q_i}{g_c} Z_i \quad i = 1, NS. \quad (2.54)$$

If the potential energy term is combined with the body force term, Eq. (2.54) may be rewritten in the form

$$e_i = i_i + \frac{q_i^2}{2} \quad i = 1, NS \quad (2.60)$$

Defining the specific enthalpy of species  $i$  as

$$h_i = i_i + \frac{P_i}{\rho_i} \quad i = 1, NS \quad (2.61)$$

where  $P_i$  = partial pressure of species  $i$   
and combining Eqs. (2.60) and (2.61) yields

$$e_i = h_i - \frac{P_i}{\rho_i} + \frac{q_i^2}{2}$$

Next, define the total enthalpy per unit mass of species  $i$  as

$$H_i = h_i + \frac{q_i^2}{2}. \quad (2.62)$$

Therefore, the expression for  $e_i$  becomes

$$e_i = H_i - \frac{P_i}{\rho_i}.$$

Substituting the expression for  $e_i$  into Eq. (2.59) results in

$$\frac{\partial}{\partial t} \sum_{i=1}^{NS} \rho_i \left( H_i - \frac{P_i}{\rho_i} \right) + \nabla \cdot \sum_{i=1}^{NS} \rho_i \left( H_i - \frac{P_i}{\rho_i} \right) \vec{q}_i - \nabla \cdot \vec{Q} + \nabla \cdot P \vec{q} - \nabla \cdot (\vec{\tau} \cdot \vec{q}) - \sum_{i=1}^{NS} \vec{q}_i \cdot \rho_i \vec{f}_i = 0 \quad (2.62a)$$

The pressure  $P$  may be expressed as the sum of the partial pressures as follows

$$P = \sum_{i=1}^{NS} P_i \quad (2.63)$$

Substituting Eq. (2.63) into Eq. (2.62a)

$$\begin{aligned} \frac{\partial}{\partial t} \sum_{i=1}^{NS} \rho_i H_i - \frac{\partial P}{\partial t} + \nabla \cdot \sum_{i=1}^{NS} \rho_i \vec{q}_i H_i - \nabla \cdot \sum_{i=1}^{NS} P_i \vec{q}_i - \nabla \cdot \vec{Q} \\ + \nabla \cdot P \vec{q} - \nabla \cdot (\bar{\tau} \cdot \vec{q}) - \sum_{i=1}^{NS} \vec{q}_i \cdot \rho_i \vec{f}_i = 0 \end{aligned}$$

and noting that

$$\nabla \cdot \sum_{i=1}^{NS} \vec{q}_i P_i = \nabla \cdot P \vec{q},$$

the above equation becomes

$$\frac{\partial}{\partial t} \sum_{i=1}^{NS} \rho_i H_i - \frac{\partial P}{\partial t} + \nabla \cdot \sum_{i=1}^{NS} \rho_i \vec{q}_i H_i - \nabla \cdot \vec{Q} - \nabla \cdot (\bar{\tau} \cdot \vec{q}) - \sum_{i=1}^{NS} \vec{q}_i \cdot \rho_i \vec{f}_i = 0 \quad (2.64)$$

From the assumption that thermodynamic local equilibrium exists, the following expressions may be written in terms of the gaseous and particle species:

$$\sum_{i=1}^{NS} \rho_i H_i = \rho H + \sum_{j=1}^{NP} \rho^j H^j \quad (2.65)$$

and,

$$\sum_{i=1}^{NS} \rho_i \vec{q}_i H_i = \rho \vec{q} H + \sum_{j=1}^{NP} \rho^j \vec{q}^j H^j \quad (2.66)$$

Substituting Eqs. (2.65) and (2.66) into Eq. (2.64)

$$\begin{aligned} \frac{\partial}{\partial t} \left( \rho H + \sum_{j=1}^{NP} \rho^j H^j \right) - \frac{\partial P}{\partial t} + \nabla \cdot \left( \rho \vec{q} H + \sum_{j=1}^{NP} \rho^j \vec{q}^j H^j \right) - \nabla \cdot \vec{Q} - \nabla \cdot (\bar{\tau} \cdot \vec{q}) \\ - \sum_{i=1}^{NS} \vec{q}_i \cdot \rho_i \vec{f}_i = 0, \end{aligned}$$

and rearranging terms yield

$$\begin{aligned} \frac{\partial}{\partial t} (\rho H) + \nabla \cdot \rho \vec{q} H + \sum_{j=1}^{NP} \left[ \frac{\partial}{\partial t} (\rho^j H^j) + \nabla \cdot \rho^j \vec{q}^j H^j \right] &= \frac{\partial P}{\partial t} + \nabla \cdot (\vec{\tau} \cdot \vec{q}) \\ &+ \sum_{i=1}^{NS} \vec{q}_i \cdot \rho_i \vec{f}_i + \nabla \cdot \vec{Q}; \end{aligned} \quad (2.67)$$

where

$$H = h + q^2/2 \quad (2.68)$$

$$H^j = h^j + (q^j)^2/2. \quad (2.69)$$

Expanding the first two terms of Eq. (2.67)

$$\rho \frac{\partial H}{\partial t} + H \frac{\partial \rho}{\partial t} + \rho \vec{q} \cdot \nabla H + H \nabla \cdot \rho \vec{q},$$

and rearranging terms yield

$$\rho \left( \frac{\partial H}{\partial t} + \vec{q} \cdot \nabla H \right) + H \left( \frac{\partial \rho}{\partial t} + \nabla \cdot \rho \vec{q} \right).$$

Recalling the gas continuity Eq. (2.19)

$$\frac{\partial \rho}{\partial t} + \nabla \cdot \rho \vec{q} = 0,$$

and applying the definition of the substantial derivative to the first term in the above expression

$$\frac{\partial H}{\partial t} + \vec{q} \cdot \nabla H = \frac{DH}{Dt},$$

the first two terms of Eq. (2.67) reduce to

$$\frac{\partial}{\partial t} (\rho H) + \nabla \cdot \rho \vec{q} H = \rho \frac{DH}{Dt}. \quad (2.70)$$

The third term of Eq. (2.67) may be manipulated in a like manner.

Expanding

$$\sum_{j=1}^{NP} \left( \rho^j \frac{\partial H^j}{\partial t} + H^j \frac{\partial \rho^j}{\partial t} + \rho^j \vec{q}^j \cdot \nabla H^j + H^j \nabla \cdot \rho^j \vec{q}^j \right),$$

and rearranging terms yield

$$\sum_{j=1}^{NP} \left[ \rho^j \left( \frac{\partial H^j}{\partial t} + \vec{q}^j \cdot \nabla H^j \right) \right] + \sum_{j=1}^{NP} \left[ H^j \left( \frac{\partial \rho^j}{\partial t} + \nabla \cdot \rho^j \vec{q}^j \right) \right].$$

Recalling the particle continuity Eq. (2.26)

$$\frac{\partial \rho^j}{\partial t} + \nabla \cdot \rho^j \vec{q}^j = 0 \quad j = 1, NP,$$

and applying the definition of the substantial derivative to the first term in the above expression

$$\frac{\partial H^j}{\partial t} + \vec{q}^j \cdot \nabla H^j = \frac{DH^j}{Dt},$$

the third term of Eq. (2.67) reduces to

$$\sum_{j=1}^{NP} \left[ \frac{\partial}{\partial t} (\rho^j H^j) + \nabla \cdot \rho^j \vec{q}^j H^j \right] = \sum_{j=1}^{NP} \rho^j \frac{DH^j}{Dt}. \quad (2.71)$$

Substituting Eqs. (2.70) and (2.71) into Eq. (2.67) results in

$$\rho \frac{DH}{Dt} + \sum_{j=1}^{NP} \rho^j \frac{DH^j}{Dt} = \frac{\partial P}{\partial t} + \nabla \cdot (\vec{\tau} \cdot \vec{q}) + \sum_{i=1}^{NS} \vec{q}_i \cdot \rho_i \vec{f}_i + \nabla \cdot \vec{Q} \quad (2.72)$$

Recall that

$$H^j = h^j + \frac{(q^j)^2}{2}. \quad (2.69)$$

Therefore,

$$\frac{DH^j}{Dt} = \frac{Dh^j}{Dt} + \frac{1}{2} \frac{D(q^j)^2}{Dt} = \frac{Dh^j}{Dt} + \frac{1}{2} \frac{D}{Dt} (\vec{q}^j \cdot \vec{q}^j)$$

or,

$$\frac{DH^j}{Dt} = \frac{Dh^j}{Dt} + \frac{1}{2} \left( \vec{q}^j \cdot \frac{D\vec{q}^j}{Dt} + \vec{q}^j \cdot \frac{D\vec{q}^j}{Dt} \right) = \frac{Dh^j}{Dt} + \frac{1}{2} \left( 2\vec{q}^j \cdot \frac{D\vec{q}^j}{Dt} \right)$$

finally,

$$\frac{DH^j}{Dt} = \frac{Dh^j}{Dt} + \vec{q}^j \cdot \frac{D\vec{q}^j}{Dt} \quad (2.73)$$

Substituting Eq. (2.73) into Eq. (2.72) results in

$$\rho \frac{DH}{Dt} + \sum_{j=1}^{NP} \rho^j \left( \frac{Dh^j}{Dt} + \vec{q}^j \cdot \frac{D\vec{q}^j}{Dt} \right) = \frac{\partial P}{\partial t} + \nabla \cdot (\vec{\tau} \cdot \vec{q}) + \sum_{i=1}^{NS} \vec{q}_i \cdot \rho_i \vec{f}_i + \nabla \cdot \vec{Q} \quad (2.74)$$

Equation (2.74) is the general global energy equation written in terms of the gaseous and particle species.

To determine the effect of particle/gas conduction and radiation, the particle energy balance for the  $j^{\text{th}}$  particle is written as follows

$$\frac{DQ^j}{Dt} = -K^j \left[ 4\pi (r^j)^2 \right] (T^j - T) - \sigma 4\pi (r^j)^2 \left[ \epsilon^j (T^j)^4 - \alpha^j T^4 \right]; \quad (2.75)$$

where

$$K^j = \frac{k N_u^j}{2r^j} = \text{particle heat transfer film coefficient.} \quad (2.76)$$

Defining the Nusselt number parameter

$$G^j = \frac{N_u^j}{(Nu)_{\text{Stokes}}} \quad (2.77)$$

where

$$(Nu)_{\text{Stokes}} = 2; \quad (2.78)$$

the particle heat transfer film coefficient may be rewritten as

$$K^j = \frac{k G^j}{r^j} . \quad (2.79)$$

Furthermore the heat quantity of the particle is defined as

$$Q^j = m^j \frac{4\pi}{3} (r^j)^3 h^j . \quad (2.80)$$

Differentiating Eq. (2.80)

$$\frac{DQ^j}{Dt} = m^j \frac{4}{3} \pi (r^j)^3 \frac{Dh^j}{Dt} ,$$

rearranging terms

$$\frac{Dh^j}{Dt} = \frac{3}{4\pi m^j (r^j)^3} \frac{DQ^j}{Dt} ,$$

and substituting the above results, with the expression for  $K^j$  back into Eq. (2.75), the particle energy balance equation becomes

$$m^j \frac{4\pi}{3} (r^j)^3 \frac{Dh^j}{Dt} = - \frac{k G^j}{r^j} \left[ 4\pi (r^j)^2 \right] (T^j - T) - \sigma 4\pi (r^j)^2 \left[ \epsilon^j (T^j)^4 - \alpha^j T^4 \right] .$$

Solving for  $Dh^j/Dt$  in the above equation

$$\frac{Dh^j}{Dt} = - \frac{k G^j}{(r^j)^2 m^j} (T^j - T) - \frac{3\sigma}{m^j r^j} \left[ \epsilon^j (T^j)^4 - \alpha^j T^4 \right] , \quad (2.81)$$

and recalling Eq. (2.46)

$$A^j = \frac{9}{2} \frac{\nu_f^j}{m^j r^j{}^2} \quad (2.46)$$

the equation for  $Dh^j/Dt$  may be written as

$$\frac{Dh^j}{Dt} = - \frac{2k}{3} \frac{G^j A^j}{\nu_f^j} (T^j - T) - \frac{3\sigma}{m^j r^j} \left[ \epsilon^j (T^j)^4 - \alpha^j T^4 \right] . \quad (2.82)$$



Equation (2.82) may be simplified by letting

$$C^j = \frac{G^j c_p}{f^j P_r} = \frac{k G^j}{\nu f^j} \quad (2.83)$$

Therefore,

$$\frac{Dh^j}{Dt} = -\frac{2}{3} A^j C^j (T^j - T) - \frac{3\sigma}{m^j r^j} \left[ \epsilon^j (T^j)^4 - \alpha^j T^4 \right]. \quad (2.84)$$

This is the general particle energy balance equation. The temperature of the particle depends upon its state, where the state may be a liquid, a liquid in the process of solidifying, or a solid. The temperature is uniquely related to the particle enthalpy by the equation of state  $T^j = f(h^j)$ , tabulated.

Substituting Eqs. (2.47) and (2.84) into the general global energy Eq. (2.74) yields

$$\begin{aligned} \rho \frac{DH}{Dt} + \sum_{j=1}^{NP} \left\{ -\frac{2}{3} \rho^j A^j C^j (T^j - T) - \frac{3\sigma}{m^j r^j} \rho^j \left[ \epsilon^j (T^j)^4 - \alpha^j T^4 \right] + \vec{q}^j \cdot \rho^j A^j \Delta \vec{q}^j \right\} \\ = \frac{\partial P}{\partial T} + \nabla \cdot (\vec{\tau} \cdot \vec{q}) + \sum_{i=1}^{NS} \vec{q}_i \cdot \rho_i \vec{f}_i + \nabla \cdot \vec{Q}. \end{aligned} \quad (2.85)$$

This is the general global energy equation with radiation effects.

Expanding the first term of the above equation

$$\rho \frac{DH}{Dt} = \rho \left( \frac{\partial H}{\partial t} + \vec{q} \cdot \nabla H \right),$$

and applying the definition for total specific enthalpy

$$H = h + \frac{q^2}{2}; \quad (2.68)$$

the first term becomes

$$\rho \frac{DH}{Dt} = \rho \frac{\partial H}{\partial t} + \rho \vec{q} \cdot \nabla h + \rho \vec{q} \cdot \frac{1}{2} \nabla (\vec{q} \cdot \vec{q}).$$

Recalling the momentum balance relationship

$$\frac{1}{2} \nabla(\vec{q} \cdot \vec{q}) = \vec{q} \times (\nabla \times \vec{q}) - \frac{1}{\rho} \sum_{j=1}^{NP} \rho^j A^j \Delta \vec{q}^j - \frac{\nabla P}{\rho}, \quad (2.52)$$

and substituting it into the above equation yields

$$\rho \frac{DH}{Dt} = \rho \frac{\partial H}{\partial t} + \rho \vec{q} \cdot \left( \nabla h - \frac{\nabla P}{\rho} \right) + \rho \vec{q} \cdot \left[ \vec{q} \times (\nabla \times \vec{q}) \right] - \vec{q} \cdot \sum_{j=1}^{NP} \rho^j A^j \Delta \vec{q}^j. \quad (2.86)$$

This expression may be further simplified by noting that for an arbitrary vector  $\vec{A}$  the following identity exists.

$$\vec{A} \cdot [\vec{A} \times (\nabla \times \vec{A})] = 0.$$

Applying the above identity to Eq. (2.86)

$$\rho \frac{DH}{Dt} = \rho \frac{\partial H}{\partial t} + \rho \vec{q} \cdot \left( \nabla h - \frac{\nabla P}{\rho} \right) - \vec{q} \cdot \sum_{j=1}^{NP} \rho^j A^j \Delta \vec{q}^j, \quad (2.86a)$$

and substituting the result back into Eq. (2.85) yields

$$\begin{aligned} \rho \frac{\partial H}{\partial t} + \rho \vec{q} \cdot \left( \nabla h - \frac{\nabla P}{\rho} \right) - \vec{q} \cdot \sum_{j=1}^{NP} \rho^j A^j \Delta \vec{q}^j + \sum_{j=1}^{NP} \left\{ -\frac{2}{3} \rho^j A^j C^j (T^j - T) \right. \\ \left. - \frac{3\sigma}{m^j r^j} \rho^j \left[ \epsilon^j (T^j)^4 - \alpha^j T^4 \right] + \vec{q}^j \cdot \rho^j A^j \Delta \vec{q}^j \right\} = \frac{\partial P}{\partial t} + \nabla \cdot (\vec{\tau} \cdot \vec{q}) \\ + \sum_{i=1}^{NS} \vec{q}_i \cdot \rho_i \vec{f}_i + \nabla \cdot \vec{Q}. \end{aligned}$$

For the case in which the flow may be described as steady state, adiabatic, inviscid and no body forces present the above equation becomes

$$\begin{aligned} \rho \vec{q} \cdot \left( \nabla h - \frac{\nabla P}{\rho} \right) - \vec{q} \cdot \sum_{j=1}^{NP} \rho^j A^j \Delta \vec{q}^j + \sum_{j=1}^{NP} \left\{ -\frac{2}{3} \rho^j A^j C^j (T^j - T) \right. \\ \left. - \frac{3\sigma}{m^j r^j} \rho^j \left[ \epsilon^j (T^j)^4 - \alpha^j T^4 \right] + \vec{q}^j \cdot \rho^j A^j \Delta \vec{q}^j \right\} = 0 \quad (2.86b) \end{aligned}$$

This equation can be reduced to a simpler form by letting

$$B^j = \vec{q} \cdot \Delta \vec{q}^j - \vec{q}^j \cdot \Delta \vec{q} + \frac{2}{3} C^j (T^j - T) + \frac{3\sigma}{A^j m^j r^j} \left[ \epsilon^j (T^j)^4 - \alpha^j T^4 \right] \quad (2.86c)$$

Therefore, the global energy equation becomes

$$\rho \vec{q} \cdot \left( \nabla h - \frac{\nabla P}{\rho} \right) - \sum_{j=1}^{NP} \rho^j A^j B^j = 0 \quad (2.87)$$

To expand the term in parenthesis in the above equation, it is necessary to examine the fundamental thermodynamic relationship for a system in which chemical reactions are occurring.

### 2.3 GASEOUS THERMODYNAMIC RELATIONS

Assumption 4 given in Section 2.1 states that the gas obeys the perfect gas law and is in chemical equilibrium, nonequilibrium or is chemically frozen. A chemically frozen gas is one in which the gas stops reacting at a given species concentration so that the molecular weight along a stream-line remains at a fixed value and the ratio of specific heats is a function of temperature only. However, for the chemically reacting case the local chemical species concentrations will change in accordance with type of chemistry assumption considered for a respective analysis.

In the high-temperature low-velocity regions of the flow field, an equilibrium chemistry calculation for the gas phase is a good assumption. Local residence times of the flow are sufficiently long for all chemical reactions to proceed to completion. However, as the flow accelerates through the nozzle and exhaust plume the local flow residence time becomes less than that required for the chemical reactions to reach completion. Significant deviations from the chemical equilibrium conditions occur and ultimately the flowfield chemistry usually approaches a frozen condition. This calculation is treated best by including the kinetics in the gasdynamic calculation, thereby avoiding the problem of choosing either an equilibrium or frozen chemical analysis.

The development of the gas thermodynamic relations and the corresponding contribution to the gasdynamic relations will consider the chemical kinetics. Appendix C addresses the equilibrium chemistry analysis for gas-particle flows. Appendix E discusses the nonequilibrium chemistry analysis. Finally, a summary of the applicable equations for chemical nonequilibrium and chemical equilibrium flow is presented in Section 3.2.3.

Consider only the gas phase portion of the flow system in which there are NS-NP different species present. If the mass fraction of each species  $i$  is constant, the specific enthalpy,  $h$ , of the gas depends only on specific entropy,  $S$ , and pressure,  $P$ . However, for a variable composition

$$h = h(S, P, X_1, X_2, \dots, X_{NS-NP})$$

and thus the total differential of  $h$  is

$$dh = \left. \frac{\partial h}{\partial S} \right|_{P, X_i} dS + \left. \frac{\partial h}{\partial P} \right|_{S, X_i} dP + \sum_{i=1}^{NS-NP} \left. \frac{\partial h}{\partial X_i} \right|_{S, P, X_j} dX_i.$$

In this expression the subscript  $X_i$  implies that the mass fractions of all species are constant during the variation in question. On the other hand, the last term in the equation is a sum of partials in each of which the  $S$  and  $P$  are constant, together with all but one of the mass fractions.

For constant mass fractions the total differential of  $h$  may be written as

$$dh = T dS + \frac{1}{\rho} dP.$$

Therefore,

$$\left. \frac{\partial h}{\partial S} \right|_{P, X_i} = T \quad \text{and} \quad \left. \frac{\partial h}{\partial P} \right|_{S, X_i} = \frac{1}{\rho}$$

By defining the chemical potential  $\mu_i$  as

$$\mu_i = \left. \frac{\partial h}{\partial X_i} \right|_{S, P, X_j} = h_i = \text{specific enthalpy of species } i$$

the fundamental thermodynamic relationship for a system in which chemical reactions are occurring may be written in the form

$$dh = T dS + \frac{dP}{\rho} + \sum_{i=1}^{NS-NP} \mu_i dX_i \quad (2.88)$$

The above equation may be rewritten as

$$\nabla h - \frac{\nabla P}{\rho} = T \nabla S + \sum_{i=1}^{NS-NP} \mu_i \nabla X_i \quad (2.89)$$

Since the pressure is a function of the gas state variables ( $\rho, S$ )

$$P = P(\rho, S), \quad (2.90)$$

the expression for  $\nabla P$  may be written

$$\nabla P = \left( \frac{\partial P}{\partial \rho} \right)_S \nabla \rho + \left( \frac{\partial P}{\partial S} \right)_\rho \nabla S.$$

Noting that

$$a^2 \equiv \left( \frac{\partial P}{\partial \rho} \right)_S \quad (2.91)$$

and,

$$\left( \frac{\partial P}{\partial S} \right)_\rho = \frac{P}{c_p - R} \quad \text{Bridgeman's Equation} \quad (2.92)$$

where  $a$ ,  $c_p$  and  $R$  are local thermodynamic equilibrium properties,  $\nabla P$  becomes

$$\nabla P = a^2 \nabla \rho + \frac{P}{c_p - R} \nabla S.$$

Rearranging terms in the above expression

$$\nabla S = \left( \frac{c_p - R}{P} \right) (\nabla P - a^2 \nabla \rho), \quad (2.93)$$

and utilizing the equation of state for an ideal gas

$$P = \rho R T, \quad (2.94)$$

Eq. (2.93) becomes:

$$\nabla S = \left( \frac{c_p - R}{\rho R T} \right) (\nabla P - a^2 \nabla \rho). \quad (2.95)$$

Multiplying through by T

$$T \nabla S = \left( \frac{c_p - R}{\rho R} \right) (\nabla P - a^2 \nabla \rho), \quad (2.96)$$

and rearranging terms yields

$$T \nabla S = \left( \frac{c_p}{R} - 1 \right) \left( \frac{\nabla P}{\rho} - \frac{a^2 \nabla \rho}{\rho} \right). \quad (2.97)$$

Substituting Eq. (2.97) into Eq. (2.89) results in

$$\nabla h - \frac{\nabla P}{\rho} = \left( \frac{c_p}{R} - 1 \right) \left( \frac{\nabla P}{\rho} - \frac{a^2 \nabla \rho}{\rho} \right) + \sum_{i=1}^{NS-NP} \mu_i \chi_i. \quad (2.98)$$

Since particle species are not considered in the chemical reactions, the summation term upper limit may be defined as

$$NG = NS - NP \quad (2.99)$$

where NG = number of gas species present.

Equation (2.98) then becomes

$$\nabla h - \frac{\nabla P}{\rho} = \left( \frac{c_p}{R} - 1 \right) \left( \frac{\nabla P}{\rho} - \frac{a^2 \nabla \rho}{\rho} \right) + \sum_{i=1}^{NG} \mu_i \nabla \chi_i. \quad (2.100)$$

Substituting Eq. (2.100) into the global energy Eq. (2.87)

$$\rho \vec{q} \cdot \left[ \left( \frac{c_p}{R} - 1 \right) \left( \frac{\nabla P}{\rho} - \frac{a^2 \nabla \rho}{\rho} \right) + \sum_{i=1}^{NG} \mu_i \nabla \chi_i \right] - \sum_{j=1}^{NP} \rho^j A^j B^j = 0 ,$$

and expanding the dot product results in

$$\left( \frac{c_p}{R} - 1 \right) (\vec{q} \cdot \nabla P - \vec{q} \cdot a^2 \nabla \rho) + \rho \sum_{i=1}^{NG} \mu_i \vec{q} \cdot \nabla \chi_i - \sum_{j=1}^{NP} \rho^j A^j B^j = 0 \quad (2.101)$$

Equation (2.101) is the expanded form of the global energy equation for flow described as steady state, adiabatic, inviscid and no body forces present.

Dividing through by  $(c_p/R - 1)$  yields

$$\vec{q} \cdot \nabla P - a^2 \vec{q} \cdot \nabla \rho + \frac{\rho}{c_p/R - 1} \sum_{i=1}^{NG} \mu_i \frac{DX_i}{Dt} - \frac{1}{c_p/R - 1} \sum_{j=1}^{NP} \rho^j A^j B^j = 0. \quad (2.102)$$

Recalling the species continuity equation

$$\rho \frac{DX_i}{Dt} = \dot{w}_i \quad i = 1, NS , \quad (2.25)$$

and letting,

$$B_1^j = \frac{B^j}{c_p/R - 1} \quad (2.103)$$

and

$$\psi_1 = \frac{1}{c_p/R - 1} \sum_{i=1}^{NG} \mu_i \dot{w}_i , \quad (2.104)$$

the final form of Eq. (2.102) becomes

$$\vec{q} \cdot \nabla P - a^2 \vec{q} \cdot \nabla \rho + \psi_1 - \sum_{j=1}^{NP} \rho^j A^j B_1^j = 0 . \quad (2.105)$$

## 2.4 SUMMARY OF THE GOVERNING EQUATIONS FOR STEADY, ADIABATIC, NONEQUILIBRIUM FLOWS OF REACTING GAS-PARTICLE MIXTURES WITHOUT TRANSPORT OR BODY FORCE EFFECTS

The system of basic governing equations for any reacting flow field have now been derived. When the reactions between components of the gas mixture are known and the boundary conditions adequately specified, one should be able to solve the nonequilibrium flow system.

The pertinent equations derived in this section are summarized as follows:

Continuity equation for steady state flow

$$\nabla \cdot \rho \vec{q} = 0 \quad \text{gas continuity equation} \quad (2.19)$$

$$\nabla \cdot \rho^j \vec{q}^j = 0 \quad j = 1, NP \quad \text{particle continuity equation} \quad (2.26)$$

$$\rho \vec{q} \cdot \nabla X_i - \dot{w}_i = 0 \quad i = 1, NS \quad \text{species continuity equation} \quad (2.25)$$

Momentum equation for steady state, inviscid flow with no body forces

$$\rho \vec{q} \cdot \nabla \vec{q} + \sum_{j=1}^{NP} \rho^j A^j \Delta \vec{q}^j + \nabla P = 0 \quad \text{global (gas + particle) momentum equation} \quad (2.50)$$

where

$$A^j = \frac{9}{2} \left[ \frac{\nu f^j}{m^j r^j{}^2} \right] \quad (2.46)$$

$$\rho^j \vec{q}^j \cdot \nabla \vec{q}^j - \rho^j A^j \nabla \vec{q}^j = 0 \quad j = 1, NP \quad \text{particle momentum equation} \quad (2.47)$$

Energy equation for steady state, inviscid flow, with no body forces and no heat loss from the gas-particle system

$$\vec{q} \cdot \nabla P - a^2 \vec{q} \cdot \nabla \rho - \sum_{j=1}^{NP} \rho^j A^j B_1^j + \psi_1 = 0 \quad \text{global energy equation} \quad (2.105)$$



where

$$B_1^j = \frac{1}{c_p/R - 1} \left\{ \vec{q} \cdot \Delta \vec{q}^j - \vec{q}^j \cdot \Delta \vec{q} + \frac{2}{3} C^j (T^j - T) + \frac{3\sigma}{A^j m^j r^j} \left[ \epsilon^j (T^j)^4 - \alpha^j T^4 \right] \right\} \quad (2.103)$$

$$\Delta \vec{q}^j = \vec{q} - \vec{q}^j \quad (2.43)$$

and

$$C^j = \frac{k G^j}{\nu f^j} \quad (2.83)$$

$$\vec{q}^j \cdot \nabla h^j + \frac{2}{3} A^j C^j (T^j - T) + \frac{3\sigma}{m^j r^j} \left[ \epsilon^j (T^j)^4 - \alpha^j T^4 \right] = 0 \quad \begin{array}{l} j = 1, NP \\ \text{particle energy} \\ \text{balance equation} \end{array} \quad (2.84)$$

### Section 3

## THE METHOD OF CHARACTERISTICS SOLUTION

The set of equations summarized in Section 2.4 are applicable for flow calculations in both the subsonic and supersonic regimes. Examination of the equations reveals that a closed form solution is not possible, thus necessitating a numerical solution to obtain the flowfield properties. The choice of the numerical solution is governed by the application and the flow regime of interest.

The current study addresses the nozzle-plume flowfield where the flow is everywhere supersonic except for those regions in the immediate vicinity downstream of a normal shock. Treatment of these regions are considered beyond the scope of this study, thus reducing the problem to one of analyzing a supersonic flowfield. For supersonic flow applications the method-of-characteristics provides a reliable method for numerically solving the set of governing equations. The method is characterized by rapid convergence of the numerical solution and has been shown to be unconditionally stable. This method is the one chosen for the current study.

A number of investigators (Refs. 1, 3 and 9) have developed solutions employing the method-of-characteristics. In "characteristic form" the relations are transformed to a set of differential equations which apply along the characteristic directions. Prozan (Ref. 9) developed a nozzle-plume solution which includes gas thermochemistry considerations. Kliegel (Ref. 3) and Hoffman (Ref. 1) extended the method-of-characteristics to include the treatment of gas-particle flows. The present study followed the work of Prozan in the treatment of the gas equilibrium thermochemistry considerations and that of Thoenes, etc., (Ref. 13) for chemical kinetics. These considerations have been combined with the coupled gas-particle solution as formulated by Kliegel.

The original formulation by Prozan utilized the enthalpy-entropy-velocity form of the compatibility equations, i.e., independent variables are the velocity, total enthalpy and entropy). Prozan showed that the thermochemistry calculations could be uncoupled from the gasdynamic solution. Consequently the thermodynamics were calculated in terms of the independent variables, V, H, S and retrieved via interpolation from tables as needed by the characteristic solution. The routines became an integral part of the stream-line normal code developed by Ruo (Ref. 10) from which the RAMP code evolved. Consequently the V, H, S form is used in the present analysis for chemical equilibrium calculations. However, when kinetics or transition between the continuum or non-continuum flow is to be treated, use of the pressure-density-velocity form of compatibility equations is more appropriate. Therefore development of both forms is given in the following sections.

### 3.1 DEVELOPMENT OF THE CHARACTERISTIC CURVES

The governing flow equations summarized in Section 2.4 may be written in the following expanded two-dimensional (or axisymmetric) form:

Continuity equation for steady state, particle species not considered in the chemical reactions

$$(2.19) \quad \rho u_x + \rho v_y + u \rho_x + v \rho_y + \frac{\delta \rho v}{y} = 0 \quad (3.1)$$

$$(2.26) \quad \rho^j u_x^j + \rho^j v_y^j + u^j \rho_x^j + v^j \rho_y^j + \frac{\delta \rho^j v^j}{y} = 0 \quad j = 1, NP \quad (3.2)$$

$$(2.25) \quad \rho u (X_i)_x + \rho v (X_i)_y - \dot{w}_i = 0 \quad i = 1, NG \quad (3.3)$$

Momentum equation for steady state, inviscid flow, with no body forces

$$(2.50) \quad \rho u u_x + \rho v u_y + P_x + \sum_{j=1}^{NP} \rho^j A^j (u - u^j) = 0 \quad (3.4)$$

$$\rho u v_x + \rho v v_y + P_y + \sum_{j=1}^{NP} \rho^j A^j (v - v^j) = 0 \quad (3.5)$$

$$(2.47) \quad \rho^j u^j u_x^j + \rho^j v^j u_y^j - \rho^j A^j (u - u^j) = 0 \quad j = 1, NP \quad (3.6)$$

$$\rho^j u^j v_x^j + \rho^j v^j v_y^j - \rho^j A^j (v - v^j) = 0 \quad j = 1, NP \quad (3.7)$$

Energy equation for steady state, inviscid flow, with no body forces and no heat loss from the gas-particle system

$$(2.105) \quad u P_x + v P_y - a^2 u \rho_x - a^2 v \rho_y - \sum_{j=1}^{NP} \rho^j A^j B_1^j + \psi_1 = 0 \quad (3.8)$$

$$(2.84) \quad \rho^j u^j h_x^j + \rho^j v^j h_y^j + \frac{2}{3} \rho^j A^j C^j (T^j - T) + \frac{3 \rho^j \sigma}{m^j r^j} [\epsilon^j (T^j)^4 - \alpha^j T^4] = 0 \quad j = 1, NP \quad (3.9)$$

The corresponding coordinate system used in this analysis is presented in Fig. 3-1.

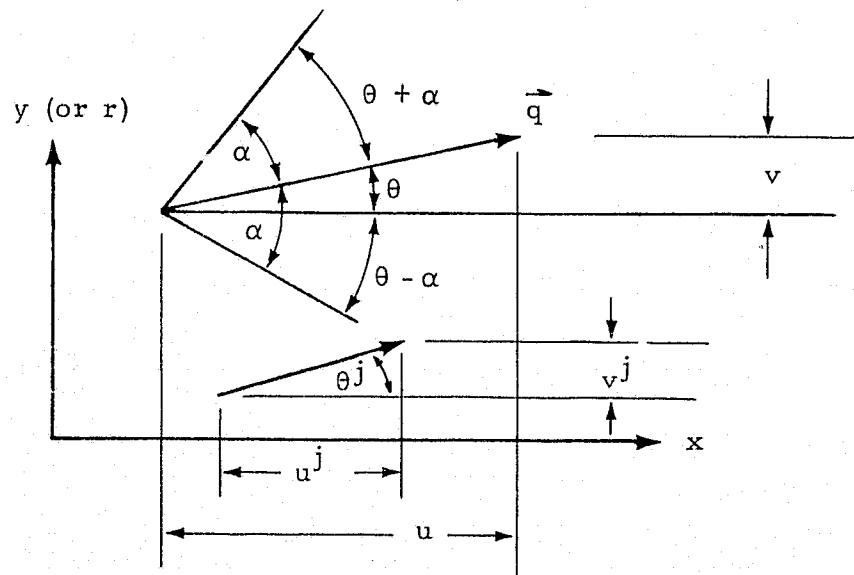


Fig. 3-1 - Coordinate System Used in the Development of the Characteristic and Compatibility Equations

To minimize the amount of "bookkeeping" that would be required in the development of the characteristic equations the governing flow Eqs. (3.1) through (3.9) will be written in the following general form

$$L_m = a_{mn} \frac{\partial \phi_n}{\partial x} + b_{mn} \frac{\partial \phi_n}{\partial y} + c_m \quad \text{for } n = 1, N \text{ and } m = 1, N \quad (3.10)$$

where:

$$N = 4 + 4NP + NG$$

$\phi_n$  represents the dependent variables  $u, v, P, \rho, u^j, v^j, \rho^j, h^j$  and  $X_i$

$a_{mn}$  and  $b_{mn}$  are coefficients.

If we let

$$\left. \begin{aligned} \phi_1 &= u \\ \phi_2 &= v \\ \phi_3 &= P \\ \phi_4 &= \rho \\ \phi_{4j+1} &= u^j \quad j = 1, NP \\ \phi_{4j+2} &= v^j \quad j = 1, NP \\ \phi_{4j+3} &= \rho^j \quad j = 1, NP \\ \phi_{4j+4} &= h^j \quad j = 1, NP \\ \phi_{4+4NP+i} &= X_i \quad i = 1, NG; \end{aligned} \right\} \quad (3.11)$$

equation (3.10) can be rewritten as

$$\begin{aligned} L_m &= a_{m1} u_x + b_{m1} u_y + a_{m2} v_x + b_{m2} v_y + a_{m3} P_x + b_{m3} P_y \\ &+ a_{m4} \rho_x + b_{m4} \rho_y + a_{m5} u_x^1 + b_{m5} u_y^1 + a_{m6} v_x^1 + b_{m6} v_y^1 \\ &+ a_{m7} \rho_x^1 + b_{m7} \rho_y^1 + a_{m8} h_x^1 + b_{m8} h_y^1 + \dots \\ &+ a_{m, 4j+1} u_x^j + b_{m, 4j+1} u_y^j + \dots + a_{m, 4j+4} h_x^j + b_{m, 4j+4} h_y^j + \dots \\ &+ a_{m, 4+4NP+i} (X_i)_x + b_{m, 4+4NP+i} (X_i)_y + \sum_{m=1}^N c_m \quad \text{for } m = 1, N \end{aligned} \quad (3.12)$$

We then have N independent, linear nonhomogeneous equations written in N unknown derivatives. The N linear equations for  $L_m$  may be combined to form a single differential operator by employing arbitrary multipliers and summing. Thus,

$$L = \sum_{m=1}^N \sigma_m L_m = 0 \quad (3.13)$$

where  $\sigma_m$  = arbitrary multipliers.

Assuming that the following relation holds,

$$a_{mn} \sigma_m \frac{dy}{dx} = b_{mn} \sigma_m \quad \text{for } n = 1, N \text{ and } m = 1, N; \quad (3.14)$$

Eq. (3.13) can be rewritten in the form of an ordinary differential equation. Substitution of Eqs. (3.10) and (3.14) into Eq. (3.13) yields

$$L = \sum_{m=1}^N (\sigma_m a_{mn} \frac{\partial \phi_n}{\partial x} + \sigma_m a_{mn} \frac{dy}{dx} \frac{\partial \phi_n}{\partial y} + c_m \sigma_m) = 0$$

or

$$L = \sum_{m=1}^N (\sigma_m a_{mn} \frac{\partial \phi_n}{\partial x} dx + \sigma_m a_{mn} \frac{\partial \phi_n}{\partial y} dy + c_m \sigma_m dx) = 0$$

and finally,

$$L = \sum_{m=1}^N (\sigma_m a_{mn} d\phi_n + c_m \sigma_m dx) = 0 \quad n = 1, N. \quad (3.15)$$

Equation (3.15) is the generalized compatibility equation and is valid if and only if Eq. (3.14) is true.

Equation (3.14) may be rewritten in the form

$$\sigma_m (a_{mn} \frac{dy}{dx} - b_{mn}) = 0 \quad n = 1, N \text{ and } m = 1, N. \quad (3.16)$$

If Eq. (3.16) has a solution other than trivial, i.e., all  $\sigma_m = 0$ , then the determinant of the coefficients of  $\sigma_m$  must be zero. Therefore:

$$D = \left| a_{mn} \frac{dy}{dx} - b_{mn} \right| = 0 \quad n = 1, N \text{ and } m = 1, N \quad (3.17)$$

The equation  $D = 0$  is called the characteristic equation for the system of equations (3.16). On expanding the determinant, it is seen that Eq. (3.17) is an algebraic equation of degree  $N$  and thus has  $N$  real roots. These roots are called the characteristic roots.

To analyze the determinant  $D$ , begin by substituting Eqs. (3.1), (3.4), (3.5) and (3.8) into Eq. (3.12), and putting the results into a matrix of the form:

$$\lambda = \begin{vmatrix} a_{11}, b_{11} & a_{12}, b_{12} & a_{13}, b_{13} & a_{14}, b_{14} \\ a_{21}, b_{21} & a_{22}, b_{22} & a_{23}, b_{23} & a_{24}, b_{24} \\ a_{31}, b_{31} & a_{32}, b_{32} & a_{33}, b_{33} & a_{34}, b_{34} \\ a_{41}, b_{41} & a_{42}, b_{42} & a_{43}, b_{43} & a_{44}, b_{44} \end{vmatrix} \quad (3.18)$$

This yields

$$\lambda_1 = \begin{vmatrix} \rho, 0 & 0, \rho & 0, 0 & u, v \\ \rho u, \rho v & 0, 0 & 1, 0 & 0, 0 \\ 0, 0 & \rho u, \rho v & 0, 1 & 0, 0 \\ 0, 0 & 0, 0 & u, v & -a^2 u, -a^2 v \end{vmatrix} \quad (3.19)$$

and

$$\begin{aligned} C_1 &= \frac{\delta \rho v}{y} \\ C_2 &= \sum_{j=1}^{NP} \rho^j A^j (u - u^j) \end{aligned} \quad (3.20)$$

$$C_3 = \sum_{j=1}^{NP} \rho^j A^j (v - v^j) \quad (3.20)$$

$$C_4 = \psi_1 - \sum_{j=1}^{NP} \rho^j A^j B_1^j \quad \text{Cont'd}$$

Next, substituting Eqs. (3.2), (3.6), (3.7) and (3.9) into Eq. (3.12) with  $j = 1$  (for the first particle) and putting the results into a matrix of the form of (3.18) yields

$$\lambda_2^1 = \begin{vmatrix} \rho^1, 0 & 0, \rho^1 & u^1, v^1 & 0, 0 \\ \rho^1 u^1, \rho^1 v^1 & 0, 0 & 0, 0 & 0, 0 \\ 0, 0 & \rho^1 u^1, \rho^1 v^1 & 0, 0 & 0, 0 \\ 0, 0 & 0, 0 & 0, 0 & \rho^1 u^1, \rho^1 v^1 \end{vmatrix} \quad (3.21)$$

and

$$\left. \begin{aligned} C_5 &= \frac{\delta \rho^1 v^1}{y} \\ C_6 &= -\rho^1 A^1 (u - u^1) \\ C_7 &= -\rho^1 A^1 (v - v^1) \\ C_8 &= \rho^1 \psi_2^1 \end{aligned} \right\} \quad (3.22)$$

where

$$\psi_2^j = \frac{2}{3} A^j C^j (T^j - T) + \frac{3\sigma}{m_r^j} \left[ \epsilon^j (T^j)^4 - \alpha^j T^4 \right]. \quad (3.23)$$

Repeating for each particle,  $j$ , and generalizing yields

$$\lambda_2^j = \begin{vmatrix} \rho^j, 0 & 0, \rho^j & u^j, v^j & 0, 0 \\ \rho^j u^j, \rho^j v^j & 0, 0 & 0, 0 & 0, 0 \\ 0, 0 & \rho^j u^j, \rho^j v^j & 0, 0 & 0, 0 \\ 0, 0 & 0, 0 & 0, 0 & \rho^j u^j, \rho^j v^j \end{vmatrix} \quad j = 1, NP \quad (3.24)$$



and

$$\left. \begin{aligned} C_{4j+1} &= \frac{\delta \rho^j v^j}{y} \\ C_{4j+2} &= -\rho^j A^j (u - u^j) \\ C_{4j+3} &= -\rho^j A^j (v - v^j) \\ C_{4j+4} &= \rho^j \psi_2^j \end{aligned} \right\} \quad j = 1, NP \quad (3.25)$$

Substituting Eq. (3.3) into Eq. (3.12)

$$\left. \begin{aligned} a_{m, 4+4NP+i} &= 0 \quad \text{if } m \neq n \quad n = 4+4NP+i \\ &= \rho u \quad \text{if } m = n \\ b_{m, 4+4NP+i} &= 0 \quad \text{if } m \neq n \\ &= \rho v \quad \text{if } m = n \end{aligned} \right\} \quad (3.26)$$

$$c_{4+4NP+i} = -\dot{w}_i \quad i = 1, NG; \quad (3.27)$$

and rewriting in the matrix form of (3.18) yields

$$\lambda_3 = \begin{vmatrix} \rho u, \rho v & 0, 0 & \dots & 0, 0 \\ 0, 0 & & & \\ \cdot & & & \\ \cdot & \rho u, \rho v & & \\ \cdot & & & 0, 0 \\ 0, 0 & 0, 0 & \dots & \rho u, \rho v \end{vmatrix} \quad (3.28)$$

If all of the coefficients of Eq. (3.12) were put into a single matrix of the form:

$$\lambda = \begin{vmatrix} a_{11}, b_{11} & a_{12}, b_{12} & \dots & a_{1N}, b_{1N} \\ a_{21}, b_{21} & & & \\ \cdot & & & \\ \cdot & & & \\ a_{N1}, b_{N1} & \dots & \dots & a_{NN}, b_{NN} \end{vmatrix}, \quad (3.29)$$

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where

$$N = 4 + 4NP + NG ;$$

the result would be the diagonal matrix

$$\lambda = \begin{vmatrix} \lambda_1 & 0 & 0 & \dots & 0 & \dots & 0 & 0 \\ 0 & \lambda_2^1 & 0 & \dots & 0 & \dots & 0 & 0 \\ 0 & 0 & \lambda_2^2 & \dots & 0 & \dots & 0 & 0 \\ 0 & 0 & 0 & \dots & 0 & \dots & 0 & 0 \\ 0 & 0 & 0 & \dots & \lambda_2^j & \dots & 0 & 0 \\ 0 & 0 & 0 & \dots & 0 & \dots & 0 & 0 \\ 0 & 0 & 0 & \dots & 0 & \dots & \lambda_2^{NP} & 0 \\ 0 & 0 & 0 & \dots & 0 & \dots & 0 & \lambda_3 \end{vmatrix} \quad (3.30)$$

Examining matrix (3.30), from which the coefficients  $a_{mn}$  and  $b_{mn}$  for determinant D will come, it can be seen that D will have the same form as matrix (3.30). That is:

$$D = \begin{vmatrix} D_1 & 0 & 0 & \dots & 0 & \dots & 0 & 0 \\ 0 & D_2^1 & 0 & \dots & 0 & \dots & 0 & 0 \\ 0 & 0 & D_2^2 & \dots & 0 & \dots & 0 & 0 \\ 0 & 0 & 0 & \dots & 0 & \dots & 0 & 0 \\ 0 & 0 & 0 & \dots & D_2^j & \dots & 0 & 0 \\ 0 & 0 & 0 & \dots & 0 & \dots & 0 & 0 \\ 0 & 0 & 0 & \dots & 0 & \dots & D_2^{NP} & 0 \\ 0 & 0 & 0 & \dots & 0 & \dots & 0 & D_3 \end{vmatrix} \quad (3.31)$$

where

$|D_1|$  obtains its elements from  $\lambda_1$

$|D_2^1|$  from  $\lambda_2^1$ , etc.

For purposes of simplification, let:

$$\left. \begin{aligned} S &= u \frac{dy}{dx} - v \\ S^j &= u^j \frac{dy}{dx} - v^j \\ y' &= \frac{dy}{dx} \end{aligned} \right\} \quad (3.32)$$

Substituting the coefficients from matrix (3.19) into matrix (3.17) and using Eq. (3.32) results in

$$|D_1| = \begin{vmatrix} \rho y' & -\rho & 0 & S \\ \rho S & 0 & y' & 0 \\ 0 & \rho S & -1 & 0 \\ 0 & 0 & S & -a^2 S \end{vmatrix} \quad (3.33)$$

Matrix (3.33) may be rewritten in the form

$$|D_1| = \sum_{j=1}^4 d_{ij} (-1)^{i+j} N_{ij} \quad (3.34)$$

where

$d_{ij}$  = elements of  $D_1$

and

$N_{ij}$  = the minor of  $d_{ij}$ .

Utilizing Eq. (3.34) and letting  $i = 3$ , Eq. (3.33) becomes

$$|D_1| = \cancel{d_{31}}^0 (-1)^4 N_{31} + d_{32} (-1)^5 N_{32} + d_{33} (-1)^6 N_{33} + \cancel{d_{34}}^0 (-1)^7 N_{34}.$$

Expanding terms

$$|D_1| = -\rho S \begin{vmatrix} \rho y' & 0 & S \\ \rho S & y' & 0 \\ 0 & S & -a^2 S \end{vmatrix} + (-1) \begin{vmatrix} \rho y' & -\rho & S \\ \rho S & 0 & 0 \\ 0 & 0 & -a^2 S \end{vmatrix};$$

performing the necessary matrix algebra

$$|D_1| = -\rho S \left[ -\rho a^2 S(y')^2 + \rho S^3 \right] - 1 (-\rho^2 a^2 S^2) ;$$

and combining terms yields

$$|D_1| = \rho^2 S^2 \left[ a^2 (y')^2 - S^2 + a^2 \right] .$$

Substituting the expressions for S and y' from Eqs. (3.32)

$$|D_1| = \rho^2 \left( u \frac{dy}{dx} - v \right)^2 \left[ a^2 \left( \frac{dy}{dx} \right)^2 - \left( u \frac{dy}{dx} - v \right)^2 + a^2 \right] ;$$

expanding

$$|D_1| = \rho^2 \left( u \frac{dy}{dx} - v \right)^2 \left[ a^2 \left( \frac{dy}{dx} \right)^2 - u^2 \left( \frac{dy}{dx} \right)^2 + 2 uv \frac{dy}{dx} - v^2 + a^2 \right] ;$$

and recombining terms yields the final form of  $|D_1|$ .

$$|D_1| = \rho^2 \left( u \frac{dy}{dx} - v \right)^2 \left[ (a^2 - u^2) \left( \frac{dy}{dx} \right)^2 + 2 uv \frac{dy}{dx} + (a^2 - v^2) \right] \quad (3.35)$$

The same procedure is applied to matrix (3.24). Substituting the coefficients from matrix (3.24) into matrix (3.17) and using Eqs. (3.32) results in

$$|D_2^j| = \begin{vmatrix} \rho^j y' & -\rho^j & S^j & 0 \\ \rho^j S^j & 0 & 0 & 0 \\ 0 & \rho^j S^j & 0 & 0 \\ 0 & 0 & 0 & \rho^j S^j \end{vmatrix} \quad j = 1, NP . \quad (3.36)$$

Utilizing Eq. (3.34) and letting  $i = 2$ , Eq. (3.36) becomes

$$|D_2^j| = d_{21} (-1)^3 N_{21} + \cancel{d_{22}^0} (-1)^4 N_{22} + \cancel{d_{23}^0} (-1)^5 N_{23} + \cancel{d_{24}^0} (-1)^6 N_{24}$$

Expanding terms, performing the necessary matrix algebra

$$|D_2^j| = \rho^j s^j (-1) \begin{vmatrix} -\rho^j & s^j & 0 \\ \rho^j s^j & 0 & 0 \\ 0 & 0 & \rho^j s^j \end{vmatrix} = -\rho^j s^j [-(\rho^j)^2 (s^j)^3],$$

and combining terms yield

$$|D_2^j| = (\rho^j)^3 (s^j)^4.$$

Substituting the expression for  $S^j$  from Eq. (3.32) yields the final form of  $|D_2^j|$

$$|D_2^j| = (\rho^j)^3 \left( u^j \frac{dy}{dx} - v^j \right)^4. \quad (3.37)$$

Finally, the same procedure is applied to matrix (3.28). Substituting the coefficients from matrix (3.28) into matrix (3.17) and using Eq. (3.32) results in

$$|D_3| = \begin{vmatrix} \rho S & 0 & \dots & 0 \\ \vdots & \rho S & & \vdots \\ 0 & \dots & 0 & \dots & \rho S \end{vmatrix} = (\rho S)^{NG}.$$

Substituting the expression for  $S$  from Eq. (3.32) yields

$$|D^3| = \rho^{NG} \left( u \frac{dy}{dx} - v \right)^{NG}. \quad (3.38)$$

Since the matrices  $|D_1|$  through  $|D_3|$  are the diagonal elements of matrix  $|D|$ , the characteristic equation for the system of Eq. (3.16) can be written as

$$|D| = |D_1| * |D_2^1| * \dots * |D_2^j| * \dots * |D_2^{NP}| * |D_3| = 0 \quad (3.39)$$

Substituting Eqs. (3.35), (3.37) and 3.38) into Eq. (3.39)

$$|D| = \rho^2 \left( u \frac{dy}{dx} - v \right)^2 \left[ (a^2 - u^2) \left( \frac{dy}{dx} \right)^2 + 2uv \frac{dy}{dx} + (a^2 - v^2) \right] \rho^{NG} \left( u \frac{dy}{dx} - v \right)^{NG}$$

$$(\rho^j)^3 \left( u^j \frac{dy}{dx} - v^j \right)^4 = 0 ,$$

and combining terms; the final form of the characteristic equation becomes:

$$|D| = \left( u \frac{dy}{dx} - v \right)^{2+NG} \left[ (a^2 - u^2) \left( \frac{dy}{dx} \right)^2 + 2uv \frac{dy}{dx} + (a^2 - v^2) \right] (\rho^j)^3 \left( u^j \frac{dy}{dx} - v^j \right)^4 = 0 \quad (3.40)$$

Realizing that  $\rho^j \neq 0$ , the characteristic roots of Eq. (3.40) are determined by setting each of the three remaining terms to zero and solving for  $dy/dx$ .

Setting the first term to zero

$$\left( u \frac{dy}{dx} - v \right)^{2+NG} = 0 ,$$

and solving for  $dy/dx$  results in

$$\frac{dy}{dx} = \frac{v}{u} = \tan \theta . \quad (3.41)$$

Setting the fourth term to zero

$$\left( u^j \frac{dy}{dx} - v^j \right)^4 = 0, \quad j = 1, NP$$

and solving for  $dy/dx$  results in

$$\frac{dy}{dx} = \frac{v^j}{u^j} = \tan \theta^j \quad j = 1, NP . \quad (3.42)$$

Referring to Fig. 3-1, Eqs. (3.41) and (3.42) shows that each of the characteristics is inclined at an angle tangent to the gas velocity vector and particle velocity vectors respectively. Thus we see that the characteristics are identical with the gas streamlines and particle streamlines of the flow.

The final characteristic root is obtained by setting the second term in Eq. (3.40) to zero

$$(a^2 - u^2) \left( \frac{dy}{dx} \right)^2 + 2uv \frac{dy}{dx} + (a^2 - v^2) = 0 ,$$

and solving for  $dy/dx$  in the following manner:

Applying the quadratic formula

$$\frac{dy}{dx} = \frac{-2uv \pm \sqrt{4u^2v^2 - 4(a^2 - u^2)(a^2 - v^2)}}{2(a^2 - u^2)} ;$$

expanding the terms under the square root

$$\frac{dy}{dx} = \frac{-2uv \pm 2\sqrt{u^2v^2 - a^4 + a^2v^2 + a^2u^2 - u^2v^2}}{2(a^2 - u^2)} ;$$

dividing through by 2 and combining terms under the square root

$$\frac{dy}{dx} = \frac{-uv \pm \sqrt{a^2(u^2 + v^2) - a^4}}{a^2 - u^2}$$

and finally moving  $a^4$  outside the square root yields

$$\frac{dy}{dx} = \frac{-uv \pm a^2 \sqrt{(u^2 + v^2)/a^2 - 1}}{a^2 - u^2} \quad (3.43)$$

The first term in the square root may be simplified by utilizing the definition of the gas Mach number

$$M = q/a$$

or

$$M^2 = \frac{q^2}{a^2} = \frac{u^2 + v^2}{a^2} . \quad (3.44)$$

Substituting Eq. (3.44) into Eq. (3.43) yields

$$\frac{dy}{dx} = \frac{-uv \pm a^2 \sqrt{M^2 - 1}}{a^2 - u^2} \quad (3.45)$$

Referring to Fig. 3-1, the following relations may be written:

$$u = q \cos \theta \quad (3.46)$$

$$v = q \sin \theta \quad (3.47)$$

$$\alpha = \sin^{-1} (1/M) = \sin^{-1} (a/q) \quad (3.48)$$

and

$$a = q \sin \alpha \quad (3.49)$$

$$\cot \alpha = \sqrt{M^2 - 1} \quad (3.50)$$

Substituting Eqs. (3.46) through (3.50) into Eq. (3.45)

$$\frac{dy}{dx} = \frac{-q^2 \cos \theta \sin \theta \pm q^2 \sin^2 \alpha \cot \alpha}{q^2 (\sin^2 \alpha - \cos^2 \theta)} ;$$

dividing through by  $q^2$

$$\frac{dy}{dx} = \frac{-\cos \theta \sin \theta \pm \sin \alpha \cos \alpha}{\sin^2 \alpha - \cos^2 \theta} ;$$

and multiplying the numerator and denominator by -1 yields

$$\frac{dy}{dx} = \frac{\cos \theta \sin \theta \mp \sin \alpha \cos \alpha}{\cos^2 \theta - \sin^2 \alpha} \quad (3.51)$$

To simplify Eq. (3.51) consider the following trigonometric identities.

$$\sin(\theta \mp \alpha) = \sin \theta \cos \alpha \mp \cos \theta \sin \alpha , \quad (3.52)$$

$$\cos(\theta \pm \alpha) = \cos \theta \cos \alpha \mp \sin \theta \sin \alpha , \quad (3.53)$$

and

$$\cos^2 \theta - \sin^2 \alpha = \cos(\theta + \alpha) \cos(\theta - \alpha) . \quad (3.54)$$

Multiplying Eq. (3.52) by (3.53)

$$\begin{aligned} \sin(\theta \mp \alpha) \cos(\theta \pm \alpha) &= \cos^2 \alpha \sin \theta \cos \theta \mp \sin^2 \theta \sin \alpha \cos \alpha \\ &\quad \mp \cos^2 \theta \sin \alpha \cos \alpha + \sin^2 \alpha \sin \theta \cos \theta \end{aligned}$$

and combining terms yield

$$\begin{aligned} \sin(\theta \mp \alpha) \cos(\theta \pm \alpha) &= (\sin^2 \alpha + \cos^2 \alpha) \sin \theta \cos \theta \\ &\quad \mp (\sin^2 \theta + \cos^2 \theta) \sin \alpha \cos \alpha ; \end{aligned}$$



or

$$\sin(\theta \mp \alpha) \cos(\theta \pm \alpha) = \cos\theta \sin\theta \mp \sin\alpha \cos\alpha. \quad (3.55)$$

By substituting Eqs. (3.54) and (3.55) into Eq. (3.51) the final form of  $dy/dx$  is obtained

$$\frac{dy}{dx} = \frac{\sin(\theta \mp \alpha) \cos(\theta \pm \alpha)}{\cos(\theta \pm \alpha) \cos(\theta - \alpha)} = \frac{\sin(\theta \mp \alpha)}{\cos(\theta \mp \alpha)};$$

or

$$\frac{dy}{dx} = \tan(\theta \mp \alpha). \quad (3.56)$$

Referring again to Fig. 3-1, Eq. (3.56) shows that each of the two characteristics is inclined at the Mach angle to the gas velocity vector. Thus we see that the characteristics are identical with the gas Mach lines of the flow.

In summary, the characteristics of the flow have been found to be the gas streamlines, the particle streamlines (one for each particle species) and the gas Mach lines.

### 3.2 DEVELOPMENT OF THE COMPATIBILITY EQUATIONS FOR GAS-PARTICLE FLOWS

The relationship assumed in Eq. (3.14)

$$a_{mn} \sigma_m \frac{dy}{dx} = b_{mn} \sigma_m \quad \text{for } n = 1, N \quad \text{and } m = 1, N$$

is valid along each of the characteristics. Therefore, we can now substitute the governing flow equations into Eq. (3.14) and knowing the conditions that exist along each characteristic, solve for the multipliers,  $\sigma_m$ . The corresponding compatibility equations that are valid along each characteristic are then derived by substituting the proper multipliers into the general compatibility equation. The compatibility equations will be first derived in the pressure-density-velocity form. This form of the equations is used by the RAMP computer program when the finite rate chemistry option is utilized. In this form, the local temperature may be solved for directly eliminating

the need for a time consuming iterative solution. The compatibility equations will then be rederived in the enthalpy-entropy-velocity form. This form of the equations is used by the RAMP computer program when the equilibrium and/or frozen chemistry option is utilized. In this form, the thermochemical data generated by the TRAN72 computer program may be used.

### 3.2.1 Pressure-Density-Velocity Form of the Compatibility Equations

Substituting the governing flow, Eqs. (3.1) through (3.9) into Eq. (3.14) results in

for  $n = 1$ :

$$\sigma_1 \rho \frac{dy}{dx} + \sigma_2 (\rho u \frac{dy}{dx} - \rho v) = 0$$

or

$$\sigma_1 \frac{dy}{dx} + \sigma_2 (u \frac{dy}{dx} - v) = 0 ; \quad (3.57)$$

for  $n = 2$ :

$$-\sigma_1 \rho + \rho (u \frac{dy}{dx} - v) \sigma_3 = 0$$

or

$$\sigma_1 = (u \frac{dy}{dx} - v) \sigma_3 ; \quad (3.58)$$

for  $n = 3$ :

$$\sigma_2 \frac{dy}{dx} - \sigma_3 + (u \frac{dy}{dx} - v) \sigma_4 = 0 ; \quad (3.59)$$

for  $n = 4$ :

$$(u \frac{dy}{dx} - v) \sigma_1 - a^2 (u \frac{dy}{dx} - v) \sigma_4 = 0 ; \quad (3.60)$$

for  $n = 4j+1$ :

$$\rho^j \frac{dy}{dx} \sigma_{4j+1} + \rho^j (u^j \frac{dy}{dx} - v^j) \sigma_{4j+2} = 0 \quad j = 1, NP;$$

or

$$\frac{dy}{dx} \sigma_{4j+1} + (u^j \frac{dy}{dx} - v^j) \sigma_{4j+2} = 0 \quad j = 1, NP ; \quad (3.61)$$

for  $n = 4j+2$ :

$$-\rho^j \sigma_{4j+1} + \rho^j (u^j \frac{dy}{dx} - v^j) \sigma_{4j+3} = 0 \quad j = 1, NP$$

or

$$\sigma_{4j+1} = (u^j \frac{dy}{dx} - v^j) \sigma_{4j+3} \quad j = 1, NP ; \quad (3.62)$$

for  $n = 4j+3$ :

$$(u^j \frac{dy}{dx} - v^j) \sigma_{4j+1} = 0 \quad j = 1, NP ; \quad (3.63)$$

for  $n = 4j+4$ :

$$\rho^j (u^j \frac{dy}{dx} - v^j) \sigma_{4j+4} = 0 \quad j = 1, NP$$

or

$$(u^j \frac{dy}{dx} - v^j) \sigma_{4j+4} = 0 \quad j = 1, NP ; \quad (3.64)$$

and finally, for  $n = 4 + 4NP + i$ :

$$\rho (u \frac{dy}{dx} - v) \sigma_{4+4NP+i} = 0 \quad i = 1, NG$$

or,

$$(u \frac{dy}{dx} - v) \sigma_{4+4NP+i} = 0 \quad i = 1, NG . \quad (3.65)$$

Now, substituting the values of  $a_{mn}$  and  $c_m$  from Eqs. (3.19), (3.20), (3.24), (3.25) and (3.28) into the generalized compatibility equation (3.15) yields

$$\begin{aligned} L = & (\rho \sigma_1 + \rho u \sigma_2) du + (\rho u \sigma_3) dv + (\sigma_2 + u \sigma_4) dP + (u \sigma_1 - a^2 u \sigma_4) dp \\ & + \sum_{j=1}^{NP} \left[ (\rho^j \sigma_{4j+1} + \rho^j u^j \sigma_{4j+2}) du^j + (\rho^j u^j \sigma_{4j+3}) dv^j + (u^j \sigma_{4j+1}) d\rho^j \right. \\ & \left. + (\rho^j u^j \sigma_{4j+4}) dh^j \right] + \sum_{i=1}^{NG} \rho u \sigma_{4+4NP+i} dX_i + \left\{ \frac{\delta \rho v}{y} \sigma_1 + \sigma_2 \sum_{j=1}^{NP} \rho^j A^j (u - u^j) \right. \\ & \left. + \sigma_3 \sum_{j=1}^{NP} \rho^j A^j (v - v^j) + \sigma_4 (\psi_1 - \sum_{j=1}^{NP} \rho^j A^j B_1^j) + \sum_{j=1}^{NP} \left[ \frac{\delta \rho^j v^j}{y} \sigma_{4j+1} \right. \right. \end{aligned} \quad (3.66)$$

$$\begin{aligned}
& - \rho^j A^j (u - u^j) \sigma_{4j+2} - \rho^j A^j (v - v^j) \sigma_{4j+3} + \rho^j \psi_2^j \sigma_{4j+4} \Big] \\
& - \sum_{i=1}^{NG} \dot{w}_i \sigma_{4+4NP+i} \Big] dx = 0 .
\end{aligned} \tag{3.66}$$

(Continued)

Equation (3.66) is the expanded form of the generalized compatibility equation and is used to determine the compatibility equations that are valid along each characteristic.

To determine the compatibility equations that are applied along gas streamlines recall that along gas streamlines

$$\frac{dy}{dx} = \frac{v}{u} . \tag{3.41}$$

Therefore, Eq. (3.57) becomes:

$$\sigma_1 \frac{dy}{dx} + \sigma_2 \left( u \frac{dy}{dx} - v \right) = 0 \tag{3.57}$$

or

$$\sigma_1 = 0 . \tag{3.67}$$

Equation (3.59) becomes

$$\sigma_2 \frac{dy}{dx} - \sigma_3 + \left( u \frac{dy}{dx} - v \right) \sigma_4 = 0 \tag{3.59}$$

or

$$\sigma_2 = \frac{dx}{dy} \sigma_3 = \frac{u}{v} \sigma_3 . \tag{3.68}$$

Along a gas streamline:

$$u^j \frac{dy}{dx} - v^j \neq 0 \quad j = 1, NP .$$

Therefore, Eq. (3.63) becomes:

$$(u^j \frac{dy}{dx} - v^j) \sigma_{4j+1} = 0 \quad j = 1, NP \quad (3.63)$$

or

$$\sigma_{4j+1} = 0 \quad j = 1, NP. \quad (3.69)$$

Similarly, Eq. (3.64) becomes

$$(u^j \frac{dy}{dx} - v^j) \sigma_{4j+4} = 0 \quad j = 1, NP \quad (3.64)$$

or

$$\sigma_{4j+4} = 0 \quad j = 1, NP. \quad (3.70)$$

Recalling Eq. (3.69), Eq. (3.62) yields

$$\overset{0}{\cancel{\sigma_{4j+1}}} = (u^j \frac{dy}{dx} - v^j) \sigma_{4j+3} = 0 \quad j = 1, NP \quad (3.62)$$

or

$$\sigma_{4j+3} = 0 \quad j = 1, NP. \quad (3.71)$$

Similarly, Eq. (3.61) yields

$$\overset{0}{\cancel{\frac{dy}{dx} \sigma_{4j+1}}} + (u^j \frac{dy}{dx} - v^j) \sigma_{4j+2} = 0 \quad j = 1, NP \quad (3.61)$$

or

$$\sigma_{4j+2} = 0 \quad j = 1, NP. \quad (3.72)$$

Substituting Eqs. (3.67) through (3.72) into Eq. (3.66)

$$\begin{aligned} L = & \frac{\rho u^2}{v} \sigma_3 du + \rho u \sigma_3 dv + \left( \frac{u}{v} \sigma_3 + u \sigma_4 \right) dP + (-a^2 u \sigma_4) d\rho \\ & + \sum_{i=1}^{NG} \rho u \sigma_{4+4NP+i} dX_i \frac{u}{v} \sum_{j=1}^{NP} \rho^j A^j (u - u^j) \sigma_3 dx + \end{aligned}$$

$$+ \sum_{j=1}^{NP} \rho^j A^j (v - v^j) \sigma_3 dx + \sigma_4 (\psi_1 - \sum_{j=1}^{NP} \rho^j A^j B_1^j) dx - \sum_{i=1}^{NG} \dot{w}_i \sigma_{4+4NP+i} dx = 0 ;$$

and rearranging terms yields

$$\begin{aligned} L = & \left\{ \frac{\rho u^2}{v} du + \rho u dv + \frac{u}{v} dP + \sum_{j=1}^{NP} \rho^j A^j \left[ \frac{u}{v} (u - u^j) + (v - v^j) \right] dx \right\} \sigma_3 \\ & + \left\{ u dP - a^2 u d\rho + (\psi_1 - \sum_{j=1}^{NP} \rho^j A^j B_1^j) dx \right\} \sigma_4 \\ & + \sum_{i=1}^{NG} \left[ \sigma_{4+4NP+i} (\rho u d\chi_i - \dot{w}_i dx) \right] = 0 . \end{aligned} \quad (3.73)$$

Since the remaining multipliers,  $\sigma$ , are arbitrary, their coefficients must be zero. Therefore, the coefficient of  $\sigma_3$  yields

$$\frac{\rho u^2}{v} du + \rho u dv + \frac{u}{v} dP + \sum_{j=1}^{NP} \rho^j A^j \left[ \frac{u}{v} (u - u^j) + (v - v^j) \right] dx = 0$$

or

$$\rho u du + \rho v dv + dP + \sum_{j=1}^{NP} \rho^j A^j \left[ (u - u^j) + \frac{v}{u} (v - v^j) \right] dx = 0 \quad (3.74)$$

Since

$$q^2 = u^2 + v^2 ;$$

$$2q dq = 2u du + 2v dv$$

or

$$q dq = u du + v dv . \quad (3.75)$$

Substituting Eq. (3.75) into Eq. (3.74), and dividing through by  $\rho$  yields:

$$q dq + \frac{dP}{\rho} + \frac{1}{\rho} \sum_{j=1}^{NP} \rho^j A^j \left[ (u - u^j) + \frac{v}{u} (v - v^j) \right] dx = 0 . \quad (3.76)$$

$$u dP - a^2 u d\rho + \psi_1 dx - \sum_{j=1}^{NP} \rho^j A^j B_1^j dx = 0$$

or

$$dP - a^2 d\rho + \frac{\psi_1}{u} dx - \frac{1}{u} \sum_{j=1}^{NP} \rho^j A^j B_1^j dx = 0 \quad (3.77)$$

and finally for the coefficient of  $\sigma_{4+4NP+i}$

$$\sum_{i=1}^{NG} (\rho u dX_i - \dot{w}_i dx) = 0$$

or

$$\rho u dX_i - \dot{w}_i dx = 0 \quad i = 1, NG \quad (3.78)$$

Equations (3.76), (3.77) and (3.78) are the compatibility equations that apply along the gas streamlines.

To determine the compatibility equations that apply along Mach lines, recall that along each Mach line

$$\frac{dy}{dx} = \tan(\theta + \alpha) \quad (3.56)$$

and,

$$\frac{dy}{dx} \neq \frac{v}{u}.$$

Equation (3.60) may be rewritten in the form:

$$(u \frac{dy}{dx} - v) (\sigma_1 - a^2 \sigma_4) = 0 \quad (3.60)$$

Therefore,

$$\sigma_1 - a^2 \sigma_4 = 0$$

or

$$\sigma_1 = a^2 \sigma_4. \quad (3.79)$$

Equation (3.57) becomes

$$\sigma_1 \frac{dy}{dx} + \sigma_2 (u \frac{dy}{dx} - v) = 0$$

$$\sigma_2 = \frac{-a^2 \sigma_4 dy/dx}{(u dy/dx - v)} \quad (3.57)$$

Using Eq. (3.32)

$$S = u \frac{dy}{dx} - v; \quad (3.32)$$

the expression for  $\sigma_2$  becomes

$$\sigma_2 = \frac{-a^2 \sigma_4 dy/dx}{S} \quad (3.80)$$

Equation (3.58) becomes

$$\sigma_1 = (u \frac{dy}{dx} - v) \sigma_3 \quad (3.58)$$

or

$$\sigma_3 = \frac{\sigma_1}{S} = \frac{a^2 \sigma_4}{S} \quad (3.81)$$

Recall that along a Mach line

$$u^j \frac{dy}{dx} - v^j \neq 0 \quad j = 1, NP.$$

Therefore, Eq. (3.63) yields

$$(u^j \frac{dy}{dx} - v^j) \sigma_{4j+1} = 0 \quad j = 1, NP \quad (3.63)$$

or

$$\sigma_{4j+1} = 0 \quad j = 1, NP. \quad (3.82)$$

Applying Eq. (3.82) to Eq. (3.61)

$$\cancel{\frac{dy}{dx} \sigma_{4j+1}}^0 + (u^j \frac{dy}{dx} - v^j) \sigma_{4j+2} = 0 \quad j = 1, NP \quad (3.61)$$

or,

$$\sigma_{4j+2} = 0 \quad j = 1, NP. \quad (3.83)$$

Similarly, Eq. (3.62) yields

$$\cancel{\sigma_{4j+1}}^0 = (u^j \frac{dy}{dx} - v^j) \sigma_{4j+3} = 0 \quad j = 1, NP \quad (3.62)$$



or,

$$\sigma_{4j+3} = 0 \quad j = 1, NP. \quad (3.84)$$

Equation (3.64) yields

$$(u^j \frac{dy}{dx} - v^j) \sigma_{4j+4} = 0 \quad j = 1, NP \quad (3.64)$$

or

$$\sigma_{4j+4} = 0 \quad j = 1, NP. \quad (3.85)$$

Finally, Eq. (3.65) yields

$$(u \frac{dy}{dx} - v) \sigma_{4+4NP+i} = 0 \quad i = 1, NG \quad (3.65)$$

or

$$\sigma_{4+4NP+i} = 0 \quad i = 1, NG. \quad (3.86)$$

Substituting Eqs. (3.79) through (3.86) into Eq. (3.66)

$$\begin{aligned} L = & \left( \rho a^2 \sigma_4 - \frac{\rho u a^2 \sigma_4}{S} \frac{dy}{dx} \right) du + \frac{\rho u a^2 \sigma_4}{S} dv + \left( - \frac{a^2 \sigma_4}{S} \frac{dy}{dx} + u \sigma_4 \right) dP \\ & + \cancel{(u a^2 \sigma_4 - a^2 u \sigma_4) d\rho} + \left\{ \frac{\delta \rho v}{y} a^2 \sigma_4 - \frac{a^2 \sigma_4}{S} \frac{dy}{dx} \sum_{j=1}^{NP} \rho^j A^j (u - u^j) \right. \\ & \left. + \frac{a^2 \sigma_4}{S} \sum_{j=1}^{NP} \rho^j A^j (v - v^j) + \sigma_4 \left( \psi_1 - \sum_{j=1}^{NP} \rho^j A^j B_1^j \right) \right\} dx = 0 ; \end{aligned}$$

and rearranging terms yields

$$\begin{aligned} L = & \left\{ \left( \rho a^2 - \frac{\rho u a^2}{S} \frac{dy}{dx} \right) du + \frac{\rho u a^2}{S} dv + \left( u - \frac{a^2}{S} \frac{dy}{dx} \right) dP \right. \\ & \left. + \left[ \frac{\delta \rho v}{y} a^2 + \frac{a^2}{S} \sum_{j=1}^{NP} \rho^j A^j \left[ (v - v^j) - (u - u^j) \frac{dy}{dx} \right] + \psi_1 - \sum_{j=1}^{NP} \rho^j A^j B_1^j \right] dx \right\} \sigma_4 = 0 \end{aligned}$$

Since the multiplier,  $\sigma_4$ , is arbitrary, the coefficient must be zero. Therefore,

$$\left( \rho a^2 - \frac{\rho u a^2}{S} \frac{dy}{dx} \right) du + \frac{\rho u a^2}{S} dv + \left( u - \frac{a^2}{S} \frac{dy}{dx} \right) dP + \frac{\delta \rho v a^2}{y} dx + \frac{a^2}{S} \sum_{j=1}^{NP} \rho^j A^j \left[ (v - v^j) - (u - u^j) \frac{dy}{dx} \right] dx + \psi_1 dx - \sum_{j=1}^{NP} \rho^j A^j B_1^j dx = 0 \quad (3.87)$$

The coefficient of  $du$  may be rewritten in the form

$$\rho a^2 - \frac{\rho u a^2}{S} \frac{dy}{dx} = \rho a^2 \left( \frac{S - u \frac{dy}{dx}}{S} \right) = \rho a^2 \left( -\frac{v}{S} \right) \quad (3.88)$$

The coefficient of  $dP$  may be rewritten in the form

$$u - \frac{a^2}{S} \frac{dy}{dx} = \frac{uS - a^2 \frac{dy}{dx}}{S} = \frac{u^2 \frac{dy}{dx} - uv - a^2 \frac{dy}{dx}}{S}$$

or

$$u - \frac{a^2}{S} \frac{dy}{dx} = \frac{(u^2 - a^2) \frac{dy}{dx} - uv}{S} \quad (3.89)$$

Recalling that

$$\frac{dy}{dx} = \frac{-uv \pm a^2 \sqrt{M^2 - 1}}{a^2 - u^2}$$

and rearranging terms

$$(a^2 - u^2) \frac{dy}{dx} = -uv \pm a^2 \sqrt{M^2 - 1} \quad ,$$

$$(u^2 - a^2) \frac{dy}{dx} = uv \mp a^2 \sqrt{M^2 - 1}$$

finally,

$$(u^2 - a^2) \frac{dy}{dx} - uv = \mp a^2 \sqrt{M^2 - 1} \quad (3.90)$$

Substituting Eq. (3.90) into Eq. (3.89) yields

$$u - \frac{a^2}{S} \frac{dy}{dx} = \frac{\bar{r} a^2 \sqrt{M^2 - 1}}{S} \quad (3.91)$$

Substituting Eqs. (3.88) and (3.91) into Eq. (3.87)

$$\begin{aligned} & - \frac{a^2}{S} \rho v du + \frac{a^2}{S} \rho u dv + \frac{a^2}{S} \sqrt{M^2 - 1} dP + \frac{\delta \rho v a^2}{y} dx \\ & + \frac{a^2}{S} \sum_{j=1}^{NP} \rho^j A^j \left[ (v - v^j) dx - (u - u^j) dy \right] + \psi_1 dx - \sum_{j=1}^{NP} \rho^j A^j B_1^j dx = 0 ; \end{aligned} \quad (3.92)$$

dividing through by  $a^2 \rho / S$  and rearranging terms yields

$$\left. \begin{aligned} & u dv - v du + \sqrt{M^2 - 1} \frac{dP}{\rho} + \left( \frac{\delta v}{y} + \frac{\psi_1}{\rho a^2} \right) S dx \\ & + \frac{1}{\rho} \sum_{j=1}^{NP} \rho^j A^j \left[ (v - v^j) dx - (u - u^j) dy - \frac{S}{a^2} B_1^j dx \right] = 0 \end{aligned} \right\} \quad (3.93)$$

Recall that

$$u = q \cos \theta , \quad (3.46)$$

$$v = q \sin \theta , \quad (3.47)$$

$$\text{and } dy/dx = \tan(\theta + \alpha) \text{ along each Mach line.} \quad (3.56)$$

Therefore,  $S dx$  may be rewritten in the form

$$S dx = \left( u \frac{dy}{dx} - v \right) dx = (q \cos \theta \tan(\theta + \alpha) - q \sin \theta) dx$$

and upon rearranging terms

$$S dx = q dx \left[ \cos \theta \frac{\sin(\theta + \alpha)}{\cos(\theta + \alpha)} - \sin \theta \frac{\cos(\theta + \alpha)}{\cos(\theta + \alpha)} \right]$$

$$S dx = \frac{q dx}{\cos(\theta + \alpha)} \left[ \cos\theta \sin(\theta + \alpha) - \sin\theta \cos(\theta + \alpha) \right]. \quad (3.94)$$

To simplify the above equation consider the trigonometric identity:

$$\sin[(\theta + \alpha) \pm \theta] = \cos\theta \sin(\theta + \alpha) \pm \sin\theta \cos(\theta + \alpha). \quad (3.95)$$

Therefore,

$$S dx = \frac{q dx}{\cos(\theta + \alpha)} \sin[(\theta + \alpha) - \theta] = q dx \frac{\sin(\alpha)}{\cos(\theta + \alpha)},$$

or

$$S dx = \frac{q \sin\alpha}{\cos(\theta + \alpha)} dx. \quad (3.96)$$

From Eqs. (3.46) and (3.47) above the following expressions can be written:

$$du = -q \sin\theta d\theta + \cos\theta dq \quad (3.97)$$

and

$$dv = q \cos\theta d\theta + \sin\theta dq. \quad (3.98)$$

Therefore, the first term in Eq. (3.93) may be written as

$$u dv - v du = q^2 \cos^2\theta d\theta + q dq \sin\theta \cos\theta + q^2 \sin^2\theta d\theta - q dq \sin\theta \cos\theta$$

or

$$u dv - v du = q^2 d\theta (\cos^2\theta + \sin^2\theta)$$

and finally,

$$u dv - v du = q^2 d\theta. \quad (3.99)$$

Substituting Eqs. (3.96), (3.99) and (3.50) into Eq. (3.93) yields the final form of the compatibility equations that apply along Mach lines

$$\begin{aligned} & q^2 d\theta + \cot\alpha \frac{dP}{\rho} + \left( \frac{\delta q \sin\theta}{y} + \frac{\psi_1}{\rho a^2} \right) \left( \frac{q \sin\alpha}{\cos(\theta + \alpha)} \right) dx \\ & + \frac{1}{\rho} \sum_{j=1}^{NP} \rho^j A^j \left[ (v - v^j) dx - (u - u^j) dy - \frac{S}{a^2} B_1^j dx \right] = 0 \end{aligned} \quad (3.100)$$

Finally, to determine the compatibility equations that apply along the particle streamlines recall that along each particle streamline

$$(u^j \frac{dy}{dx} - v^j)^4 = 0 ,$$

or

$$\frac{dy}{dx} = \frac{v^j}{u^j} = \tan \theta^j \quad (3.42)$$

and that,

$$\frac{dy}{dx} \neq \frac{v}{u} .$$

We may immediately rewrite Eqs. (3.79), (3.80) and (3.81) which were derived for Mach lines.

Equation (3.79)

$$\sigma_1 = a^2 \sigma_4 , \quad (3.101)$$

Equation (3.80)

$$\sigma_2 = - \frac{a^2 \sigma_4}{S} \frac{dy}{dx} , \quad (3.102)$$

Equation (3.81)

$$\sigma_3 = \frac{a^2 \sigma_4}{S} . \quad (3.103)$$

Then applying Eq. (3.42) to Eq. (3.61) yields

$$\frac{dy}{dx} \sigma_{4j+1} + \left( u^j \frac{dy}{dx} - v^j \right) \sigma_{4j+2} = 0 \quad j = 1, NP , \quad (3.61)$$

$$\frac{dy}{dx} \sigma_{4j+1} = 0 \quad j = 1, NP$$

or

$$\sigma_{4j+1} = 0 \quad j = 1, NP . \quad (3.104)$$

Similarly, Eq. (3.65) yields

$$\left( u \frac{dy}{dx} - v \right) \sigma_{4+4NP+i} = 0 \quad i = 1, NG \quad (3.65)$$

or,

$$\sigma_{4+4NP+i} = 0 \quad i = 1, NG . \quad (3.105)$$

Substituting Eqs. (3.101) through (3.105) into Eq. (3.66)

$$\begin{aligned} L = & \left( \rho a^2 \sigma_4 - \frac{\rho u a^2}{S} \sigma_4 \frac{dy}{dx} \right) du + \frac{\rho u a^2}{S} \sigma_4 dv + \left( u \sigma_4 - \frac{a^2 \sigma_4}{S} \frac{dy}{dx} \right) dP \\ & + (u a^2 \sigma_4 - u a^2 \sigma_4) d\rho + \sum_{j=1}^{NP} \left[ \rho^j u^j \sigma_{4j+2} du^j + \rho^j u^j \sigma_{4j+3} dv^j \right. \\ & \left. + \rho^j u^j \sigma_{4j+4} dh^j \right] + \frac{\delta \rho v a^2 \sigma_4}{y} dx - \frac{a^2 \sigma_4}{S} \frac{dy}{dx} \sum_{j=1}^{NP} \rho^j A^j (u - u^j) dx \\ & + \frac{a^2 \sigma_4}{S} \sum_{j=1}^{NP} \rho^j A^j (v - v^j) dx + \sigma_4 \psi_1 dx - \sigma_4 \sum_{j=1}^{NP} \rho^j A^j B_1^j dx \\ & + \sum_{j=1}^{NP} \left[ -\rho^j A^j (u - u^j) \sigma_{4j+2} - \rho^j A^j (v - v^j) \sigma_{4j+3} + \rho^j \psi_2^j \sigma_{4j+4} \right] dx = 0 ; \end{aligned}$$

and rearranging terms yields:

$$\begin{aligned} L = & \left\{ \rho a^2 \left( 1 - \frac{u}{S} \frac{dy}{dx} \right) du + \rho a^2 \frac{u}{S} dv + \left( u - \frac{a^2}{S} \frac{dy}{dx} \right) dP + \frac{\delta \rho v a^2}{y} dx \right. \\ & + \frac{a^2}{S} \sum_{j=1}^{NP} \rho^j A^j \left[ (v - v^j) dx - (u - u^j) dy - \frac{S}{a^2} B_1^j dx \right] + \psi_1 dx \Big\} \sigma_4 \\ & + \sum_{j=1}^{NP} \left[ \rho^j u^j du^j - \rho^j A^j (u - u^j) dx \right] \sigma_{4j+2} \\ & + \sum_{j=1}^{NP} \left[ \rho^j u^j dv^j - \rho^j A^j (v - v^j) dx \right] \sigma_{4j+3} \\ & + \sum_{j=1}^{NP} \left[ \rho^j u^j dh^j + \rho^j \psi_2^j dx \right] \sigma_{4j+4} = 0 . \quad (3.106) \end{aligned}$$

Since the remaining multipliers,  $\sigma$ , are arbitrary their coefficients must be zero. Therefore, the coefficient of  $\sigma_4$  yields

$$\rho a^2 \left(1 - \frac{u}{S} \frac{dy}{dx}\right) du + \rho a^2 \frac{u}{S} dv + \left(u - \frac{a^2}{S} \frac{dy}{dx}\right) dP + \frac{\delta \rho v a^2}{y} dx + \frac{a^2}{S} \sum_{j=1}^{NP} \rho^j A^j \left[(v - v^j) dx - (u - u^j) dy - \frac{S}{a^2} B_1^j dx\right] + \psi_1 dx = 0; \quad (3.107)$$

the coefficient of  $\sigma_{4j+2}$  yields

$$\sum_{j=1}^{NP} \left[ \rho^j u^j du^j - \rho^j A^j (u - u^j) dx \right] = 0;$$

or

$$u^j du^j = A^j (u - u^j) dx \quad j = 1, NP. \quad (3.108)$$

Similarly, for the coefficient of  $\sigma_{4j+3}$

$$\sum_{j=1}^{NP} \left[ \rho^j u^j dv^j - \rho^j A^j (v - v^j) dx \right] = 0$$

or

$$u^j dv^j = A^j (v - v^j) dx \quad j = 1, NP; \quad (3.109)$$

and finally for the coefficient of  $\sigma_{4j+4}$

$$\sum_{j=1}^{NP} \left[ \rho^j u^j dh^j + \rho^j \psi_2^j dx \right] = 0$$

or

$$u^j dh^j = -\psi_2^j dx \quad j = 1, NP. \quad (3.110)$$

Recalling that

$$\psi_2^j = \frac{2}{3} A^j C^j (T^j - T) + \frac{3\sigma}{m^j r^j} \left[ \epsilon^j (T^j)^4 - \alpha^j T^4 \right]; \quad (3.23)$$

Eq. (3.110) becomes

$$u^j dh^j = - \left[ \frac{2}{3} A^j C^j (T^j - T) + \frac{3\sigma}{m^j r^j} \left[ \epsilon^j (T^j)^4 - \alpha^j T^4 \right] \right] dx \quad j = 1, NP \quad (3.111)$$

Before evaluating Eq. (3.107) it is necessary to examine Eqs. (3.57) through (3.60) with  $S = u dy/dx - v$  non-vanishing ( $S \neq 0$ ) as in the case along a particle streamline.

Equation (3.60):

$$S \sigma_1 - a^2 S \sigma_4 = 0$$

or

$$\sigma_1 = a^2 \sigma_4 ; \quad (3.112)$$

Equation (3.58):

$$\sigma_1 = S \sigma_3$$

or

$$\sigma_3 = \sigma_1 / S$$

finally,

$$\sigma_3 = \frac{a^2 \sigma_4}{S} ; \quad (3.113)$$

Equation (3.57):

$$\sigma_1 \frac{dy}{dx} + \sigma_2 S = 0$$

or

$$\sigma_2 = - \frac{\sigma_1}{S} \frac{dy}{dx} = - \frac{a^2 \sigma_4}{S} \frac{dy}{dx} ; \quad (3.114)$$

and finally

Equation (3.59):

$$\sigma_2 \frac{dy}{dx} - \sigma_3 + S \sigma_4 = 0 . \quad (3.59)$$



Substituting Eqs. (3.112) through (3.114) into Eq. (3.59) yields

$$-\frac{a^2 \sigma_4}{S} \left( \frac{dy}{dx} \right)^2 - \frac{a^2 \sigma_4}{S} + S \sigma_4 = 0$$

or

$$\left[ -a^2 \left( \frac{dy}{dx} \right)^2 - a^2 + S^2 \right] \sigma_4 = 0 \quad (3.115)$$

In the above equation, either  $\sigma_4$  is zero or its coefficient is zero.

Along particle streamlines:

$$\frac{dy}{dx} = \frac{v^j}{u^j} \quad (3.42)$$

Assuming  $\sigma_4 = 0$ , Eq. (3.115) becomes

$$\left[ -a^2 \left( \frac{v^j}{u^j} \right)^2 - a^2 + S^2 \right] = 0.$$

Recalling that

$$S = u \frac{dy}{dx} - v$$

or

$$S = u \frac{v^j}{u^j} - v \neq 0;$$

Eq. (3.115) may again be rewritten in the form:

$$-a^2 \left( \frac{v^j}{u^j} \right)^2 - a^2 + u^2 \left( \frac{v^j}{u^j} \right)^2 - 2uv \frac{v^j}{u^j} + v^2 = 0$$

or

$$(u^2 - a^2) \left( \frac{v^j}{u^j} \right)^2 - 2uv \frac{v^j}{u^j} + (v^2 - a^2) = 0$$

Solving for  $\frac{v^j}{u^j}$ ,

$$\frac{v^j}{u^j} = \frac{2uv \pm \sqrt{4u^2v^2 - 4(u^2 - a^2)(v^2 - a^2)}}{2(u^2 - a^2)}$$

upon rearranging terms,

$$\frac{v^j}{u^j} = \frac{uv \pm \sqrt{u^2 v^2 - u^2 v^2 + a^2 v^2 + a^2 u^2 - a^4}}{u^2 - a^2}$$

$$\frac{v^j}{u^j} = \frac{uv \pm \sqrt{a^2 (u^2 + v^2) - a^4}}{u^2 - a^2}$$

$$\frac{v^j}{u^j} = \frac{uv \pm a^2 \sqrt{(u^2 + v^2)/a^2 - 1}}{u^2 - a^2}$$

$$\frac{v^j}{u^j} = \frac{-uv \pm a^2 \sqrt{(u^2 + v^2)/a^2 - 1}}{a^2 - u^2}$$

This equation has the same solution as Eq. (3.43), that is:

$$\frac{v^j}{u^j} = \tan(\theta + \alpha).$$

However, for a particle streamline,

$$\frac{v^j}{u^j} = \tan \theta^j ;$$

therefore the assumption that the coefficient of  $\sigma_4$  in Eq. (3.115) vanishes does not apply along particle streamlines, hence the multiplier  $\sigma_4$  must be equal to zero and Eq. (3.107) is not valid along particle streamlines. In summary, the compatibility equations that apply along particle streamlines are:

$$\frac{dy}{dx} = \frac{v^j}{u^j} = \tan \theta^j \quad j = 1, NP$$

$$u^j du^j = A^j (u - u^j) dx \quad j = 1, NP \quad (3.108)$$

$$u^j dv^j = A^j (v - v^j) dx \quad j = 1, NP \quad (3.109)$$

and

$$u^j dh^j = - \left[ \frac{2}{3} A^j C^j (T^j - T) + \frac{3\sigma}{m^j r^j} \left[ e^{j(T^j)^4} - \alpha^j T^4 \right] \right] dx \quad j = 1, NP \quad (3-111)$$

### 3.2.2 Enthalpy-Entropy-Velocity Form of the Compatibility Equations

Recall the pressure-density-velocity form of the compatibility equations that apply along gas streamlines

$$q dq + \frac{dP}{\rho} + \frac{1}{\rho} \sum_{j=1}^{NP} \rho^j A^j \left[ (u - u^j) + \frac{v}{u} (v - v^j) \right] dx = 0 \quad (3.76)$$

$$dP - a^2 d\rho + \frac{\psi}{u} dx - \frac{1}{u} \sum_{j=1}^{NP} \rho^j A^j B_1^j dx = 0 \quad (3.77)$$

$$\rho u dX_i - \dot{w}_i dx = 0 \quad i = 1, NG \quad (3.78)$$

Equation (3.76) may be written in the enthalpy-entropy form using the thermodynamic relations

$$T dS = dh - \frac{dP}{\rho} - \sum_{i=1}^{NG} \mu_i dX_i \quad (2.88)$$

$$H = h + \frac{q^2}{2} \quad (2.68)$$

and the definition of the local speed of sound.

From Ref. 2 it is shown that since the particle velocity and temperature do not change through a discontinuity and since there is a finite relaxation time associated with changes in the particle properties, then an infinitesimal weak disturbance must travel at the gas-sonic speed. That is to say, the local speed of sound of the gas-particle mixture is unaffected by the particle and is identical with the speed of sound of the gas; hence, it can be expressed by the following

$$a^2 = \gamma R T .$$

From Eq. (2.68) we may write

$$dH = dh + q dq$$

or

$$dh = dH - q dq \quad (3.116)$$

Rearranging Eq. (2.88)

$$\frac{dP}{\rho} = dh - Tds - \sum_{i=1}^{NG} \mu_i dX_i,$$

substituting for  $dh$  from Eq. (3.116)

$$\frac{dP}{\rho} = dH - q dq - TdS - \sum_{i=1}^{NG} \mu_i dX_i$$

and rearranging terms yields

$$\frac{dP}{\rho} + q dq = dH - TdS - \sum_{i=1}^{NG} \mu_i dX_i. \quad (3.117)$$

Substituting this result back into Eq. (3.76) yields the final form of the compatibility equation

$$dH - TdS - \sum_{i=1}^{NG} \mu_i dX_i + \frac{1}{\rho} \sum_{j=1}^{NP} \rho^j A^j \left[ (u - u^j) + \frac{v}{u} (v - v^j) \right] dx = 0 \quad (3.118)$$

Equation (3.118) may be rewritten using the definition of the local speed of sound.

Solving for temperature

$$T = \frac{q^2 \sin^2 \alpha}{\gamma R}, \quad (3.119)$$

and substituting the result back into Eq. (3.118) yields

$$dH - q^2 \frac{\sin^2 \alpha}{\gamma R} dS - \sum_{i=1}^{NG} \mu_i dX_i + \frac{1}{\rho} \sum_{j=1}^{NP} \rho^j A^j \left[ (u - u^j) + \frac{v}{u} (v - v^j) \right] dx = 0 \quad (3.120)$$

Next, we examine Eq. (3.78).

Equation (3.78) may be rewritten in the form

$$\rho q \cos \theta dX_i = \dot{w}_i dx \quad i = 1, NG \quad (3.121)$$

or

$$dX_i = \frac{\dot{w}_i dx}{\rho q \cos \theta} \quad i = 1, NG \quad (3.122)$$

By letting

$$\dot{X}_i = \frac{\dot{w}_i}{\rho},$$

Eq. (3.122) becomes

$$dX_i = \frac{\dot{X}_i dx}{q \cos \theta} \quad i = 1, NG$$

or finally,

$$q \cos \theta dX_i = \dot{X}_i dx \quad i = 1, NG. \quad (3.123)$$

Equation (3.77) may be written in the enthalpy-entropy form using the thermodynamic relations

$$T dS = \left( \frac{C_p}{R} - 1 \right) \left( \frac{dP}{\rho} - \frac{a^2 d\rho}{\rho} \right) \quad (2.97)$$

and

$$\psi_1 = \frac{1}{C_p/R - 1} \sum_{i=1}^{NG} \mu_i \dot{w}_i \quad (2.104)$$

Rearranging Eq. (2.97)

$$dP - a^2 d\rho = \frac{\rho T dS}{(C_p/R - 1)}, \quad (3.124)$$

and substituting for  $T$  from Eq. (3.119) yields

$$dP - a^2 d\rho = \frac{\rho q^2 \sin^2 \alpha dS}{\gamma (C_p - R)} \quad (3.125)$$

Replacing the first two terms of Eq. (3.77) with Eq. (3.125)

$$\frac{\rho q^2 \sin^2 \alpha dS}{\gamma (C_p - R)} + \frac{\psi_1}{u} dx - \frac{1}{u} \sum_{j=1}^{NP} \rho^j A^j B_1^j dx = 0 ;$$

substituting for  $\psi_1$  from Eq. (2.104)

$$\frac{\rho q^2 \sin^2 \alpha dS}{\gamma (C_p - R)} + \frac{R}{(C_p - R)u} \sum_{i=1}^{NG} \mu_i \dot{w}_i dx - \frac{1}{u} \sum_{j=1}^{NP} \rho^j A^j B_1^j dx = 0 \quad (3.126)$$

and using the relation  $\dot{X}_i = \dot{w}_i / \rho$  results in

$$\frac{\rho q^2 \sin^2 \alpha dS}{\gamma (C_p - R)} + \frac{R\rho}{(C_p - R)q \cos \theta} \sum_{i=1}^{NG} \mu_i \dot{X}_i dx - \frac{1}{u} \sum_{j=1}^{NP} \rho^j A^j B_1^j dx = 0.$$

Dividing through by  $\rho R / (C_p - R)$ , and recalling Eq. (3.119) yields

$$T dS + \frac{1}{q \cos \theta} \sum_{i=1}^{NG} \mu_i \dot{X}_i dx - \frac{(C_p - R)}{u \rho R} \sum_{j=1}^{NP} \rho^j A^j B_1^j dx = 0 \quad (3.127)$$

where

$$B_1^j = \frac{R}{C_p - R} \left\{ \vec{q} \cdot \Delta \vec{q}^j - \vec{q}^j \cdot \Delta \vec{q} + \frac{2}{3} C^j (T^j - T) + \frac{3\sigma}{A_j m_j r_j} \left[ \epsilon^j (T^j)^4 - \alpha^j T^4 \right] \right\} \quad (2.103)$$

The final form of the compatibility equations that apply along gas streamlines are summarized as follows:

Equation (3.127)

$$T dS + \frac{1}{q \cos \theta} \sum_{i=1}^{NG} \mu_i \dot{X}_i dx - \frac{(C_p - R)}{u \rho R} \sum_{j=1}^{NP} \rho^j A^j B_1^j dx = 0, \quad (3.128)$$

Equation (3.118)

$$dH - T dS - \sum_{i=1}^{NG} \mu_i dX_i + \frac{1}{\rho} \sum_{j=1}^{NP} \rho^j A^j \left[ (u - u^j) + \frac{v}{u} (v - v^j) \right] dx = 0 \quad (3.129)$$

and

Equation (3.123)

$$q \cos \theta dX_i = \dot{X}_i dx \quad i = 1, NG \quad (3.130)$$

The pressure-density-velocity form of the compatibility equations that apply along Mach lines are

$$q^2 d\theta \mp \cot \alpha \frac{dP}{\rho} + \left( \frac{\delta q \sin \theta}{y} + \frac{\psi_1}{\rho a^2} \right) \left( \mp \frac{q \sin \alpha}{\cos(\theta \mp \alpha)} \right) dx + \frac{1}{\rho} \sum_{j=1}^{NP} \rho^j A^j \left[ (v - v^j) dx - (u - u^j) dy - \frac{S}{a^2} B_1^j dx \right] = 0. \quad (3.100)$$

Substituting Eq. (3.117)

$$\frac{dP}{\rho} = dH - T dS - q dq - \sum_{i=1}^{NG} \mu_i dX_i \quad (3.117)$$

into Eq. (3.100)

$$q^2 d\theta \mp \cot \alpha (dH - T dS - q dq - \sum_{i=1}^{NG} \mu_i dX_i) \mp \frac{q \sin \alpha}{\cos(\theta \mp \alpha)} \left( \frac{\delta q \sin \theta}{y} + \frac{\psi_1}{\rho a^2} \right) dx + \frac{1}{\rho} \sum_{j=1}^{NP} \rho^j A^j \left[ (v - v^j) dx - (u - u^j) dy \pm \frac{q \sin \alpha}{\cos(\theta \mp \alpha) q^2 \sin^2 \alpha} B_1^j dx \right] = 0 ;$$

dividing through by  $q^2$ , and expanding wherever possible yields:

$$\begin{aligned} d\theta + \frac{\cot\alpha}{q^2} dH + \frac{\cot\alpha}{q^2} T dS + \frac{\cot\alpha}{q} dq + \frac{\cot\alpha}{q^2} \sum_{i=1}^{NG} \mu_i dX_i \\ + \frac{\delta \sin\alpha \sin\theta}{\cos(\theta + \alpha) y} dx + \frac{\sin\alpha \psi_1}{q \rho a^2 \cos(\theta + \alpha)} dx + \frac{1}{\rho q^2} \sum_{j=1}^{NP} \rho^j A^j \left[ (v - v^j) dx - (u - u^j) dy \right. \\ \left. + \frac{B_1^j dx}{q \sin\alpha \cos(\theta + \alpha)} \right] = 0 \end{aligned} \quad (3.131)$$

Equation (3.131) may be rewritten in the following manner:

Recall that

$$T = \frac{q^2 \sin^2 \alpha}{\gamma R}, \quad (3.119)$$

$$\psi_1 = \frac{1}{C_p/R - 1} \sum_{i=1}^{NG} \mu_i \dot{w}_i \quad (3.104)$$

and

$$\dot{X}_i = \frac{\dot{w}_i}{\rho}.$$

Therefore, the third term of Eq. (3.131) may be rewritten as

$$\frac{\cot\alpha}{q^2} T dS = \frac{\cot\alpha q^2 \sin^2 \alpha}{q^2 \gamma R} dS$$

or

$$\frac{\cot\alpha}{q^2} T dS = \frac{\sin\alpha \cos\alpha}{\gamma R} dS. \quad (3.132)$$

The seventh term may be rewritten as

$$\frac{\sin\alpha \psi_1 dx}{q \rho a^2 \cos(\theta + \alpha)} = \frac{\sin\alpha \left[ \frac{1}{(C_p/R - 1)} \sum_{i=1}^{NG} \mu_i \dot{w}_i \right] dx}{\rho q^3 \sin^2 \alpha \cos(\theta + \alpha)} \quad (3.133)$$



or

$$\frac{\sin \alpha \psi_1 dx}{q \rho a^2 \cos(\theta + \alpha)} = \frac{\frac{dx}{C_p/R - 1} \sum_{i=1}^{NG} \mu_i \dot{X}_i}{q^3 \sin \alpha \cos(\theta + \alpha)} \quad (3.133)$$

A portion of the eighth term may be rewritten as

$$\begin{aligned} (v - v^j) dx - (u - u^j) dy &= dx \left[ (v - v^j) - (u - u^j) \frac{dy}{dx} \right] \\ &= dx \left[ (v - v^j) - (u - u^j) \tan(\theta + \alpha) \right] \\ &= dx \left[ (v - v^j) \frac{\cos(\theta + \alpha)}{\cos(\theta + \alpha)} - (u - u^j) \frac{\sin(\theta + \alpha)}{\cos(\theta + \alpha)} \right] \end{aligned}$$

finally,

$$(v - v^j) dx - (u - u^j) dy = \frac{dx}{\cos(\theta + \alpha)} \left[ (v - v^j) \cos(\theta + \alpha) - (u - u^j) \sin(\theta + \alpha) \right] \quad (3.134)$$

Substituting Eqs. (3.132), (3.133) and (3.134) back into Eq. (3.131) yields the final form of the compatibility equations that apply along Mach lines

$$\begin{aligned} d\theta \pm \frac{\cot \alpha}{q} dq \pm \frac{\sin \alpha \cos \alpha dS}{\gamma R} \mp \frac{\cot \alpha dH}{q^2} \mp \frac{\delta \sin \theta \sin \alpha dx}{y \cos(\theta + \alpha)} \\ \pm \frac{dx}{\rho q^2 \cos(\theta + \alpha)} \sum_{j=1}^{NP} \rho^j A^j \left[ \pm (v - v^j) \cos(\theta + \alpha) \mp (u - u^j) \sin(\theta + \alpha) + \frac{B_1^j}{q \sin \alpha} \right] \\ \pm \frac{\cot \alpha}{q^2} \sum_{i=1}^{NG} \mu_i dX_i \mp \frac{\frac{dx}{C_p/R - 1} \sum_{i=1}^{NG} \mu_i \dot{X}_i}{q^3 \sin \alpha \cos(\theta + \alpha)} = 0 \end{aligned} \quad (3.135)$$

### 3.2.3 Compatibility Equations for Equilibrium and/or Frozen Chemistry

Recall the enthalpy-entropy-velocity form of the compatibility equations.

Along gas streamlines:

$$T dS + \frac{1}{q \cos \theta} \sum_{i=1}^{NG} \mu_i \dot{X}_i dx - \frac{(C_p - R)}{u \rho R} \sum_{j=1}^{NP} \rho^j A^j B_1^j dx = 0 \quad (3.128)$$

and

$$dH - T dS - \sum_{i=1}^{NG} \mu_i dX_i + \frac{1}{\rho} \sum_{j=1}^{NP} \rho^j A^j \left[ (u - u^j) + \frac{v}{u} (v - v^j) \right] dx = 0. \quad (3.129)$$

Along Mach lines:

$$\begin{aligned} d\theta \pm \frac{\cot \alpha}{q} dq \pm \frac{\sin \alpha \cos \alpha}{\gamma R} \frac{dS}{q^2} \mp \frac{\cot \alpha}{q^2} \frac{dH}{\gamma \cos(\theta \mp \alpha)} \mp \frac{\delta \sin \theta \sin \alpha}{y \cos(\theta \mp \alpha)} dx \\ \pm \frac{dx}{\rho q^2 \cos(\theta \mp \alpha)} \sum_{j=1}^{NP} \rho^j A^j \left[ \pm (v - v^j) \cos(\theta \mp \alpha) \mp (u - u^j) \sin(\theta \mp \alpha) + \frac{B_1^j}{q \sin \alpha} \right] \\ \pm \frac{\cot \alpha}{q^2} \sum_{i=1}^{NG} \mu_i dX_i \mp \frac{\frac{dx}{C_p/R - 1} \sum_{i=1}^{NG} \mu_i \dot{X}_i}{q^3 \sin \alpha \cos(\theta \mp \alpha)} = 0 \end{aligned} \quad (3.135)$$

For any closed system which is in the state of complete equilibrium the terms

$$\sum_{i=1}^{NG} \mu_i dX_i$$

and

$$\sum_{i=1}^{NG} \mu_i \dot{X}_i$$

must be zero (Ref. 14). Furthermore, for chemically frozen flow  $\dot{X}_i$  and  $dX_i$  are obviously zero since there are no changes in the chemical species concentration. Therefore, for the case of equilibrium and/or frozen chemistry the above equations reduce to the form:

Along gas streamlines:

$$T dS - \frac{(C_p - R)}{u \rho R} \sum_{j=1}^{NP} \rho^j A^j B_1^j dx = 0, \quad (3.128a)$$

and

$$dH - T dS + \frac{1}{\rho} \sum_{j=1}^{NP} \rho^j A^j \left[ (u - u^j) + \frac{v}{u} (v - v^j) \right] dx = 0. \quad (3.129a)$$

Along Mach lines:

$$\begin{aligned} d\theta \pm \frac{\cot \alpha}{q} dq \pm \frac{\sin \alpha \cos \alpha}{\gamma R} \frac{dS}{dx} \pm \frac{\cot \alpha}{q^2} dH \pm \frac{\delta \sin \theta \sin \alpha}{y \cos(\theta \mp \alpha)} dx \\ \pm \frac{dx}{\rho q^2 \cos(\theta \mp \alpha)} \sum_{j=1}^{NP} \rho^j A^j \left[ \pm (v - v^j) \cos(\theta \mp \alpha) \mp (u - u^j) \sin(\theta \mp \alpha) + \frac{B_1^j}{q \sin \alpha} \right] \end{aligned} \quad (3.135a)$$

### 3.2.4 Summary of the Compatibility Equations for Gas-Particle Flows

The characteristics of the flow have been found to be the gas streamlines, gas Mach lines and particle streamlines (one for each particle species).

#### Pressure-Density-Velocity Form (for Chemical Non-Equilibrium and Transition Flow Applications)

The slope of the gas streamline,  $\theta$ , is given by

$$\frac{dy}{dx} = \tan \theta \quad (3.41)$$

and the compatibility equations which apply along gas streamlines are:

$$q dq + \frac{dP}{\rho} + \frac{1}{\rho} \sum_{j=1}^{NP} \rho^j A^j \left[ (u - u^j) + \frac{v}{u} (v - v^j) \right] dx = 0, \quad (3.76)$$

$$dP - a^2 d\rho + \frac{\psi}{u} dx - \frac{1}{u} \sum_{j=1}^{NP} \rho^j A^j B_1^j dx = 0 \quad (3.77)$$

and

$$\rho u dX_i - \dot{w}_i dx = 0 \quad i = 1, NG \quad (3.78)$$

The slope of the Mach lines (left running characteristics and right running characteristics) is given by

$$\frac{dy}{dx} = \tan(\theta \mp \alpha) \quad (3.56)$$

and the compatibility equations which apply along each Mach line are:

$$q^2 d\theta \mp \cot \alpha \frac{dP}{\rho} + \left( \frac{\delta q \sin \theta}{y} + \frac{\psi_1}{\rho a^2} \right) \left( \mp \frac{q \sin \alpha}{\cos(\theta \mp \alpha)} \right) dx \\ + \frac{1}{\rho} \sum_{j=1}^{NP} \rho^j A^j \left[ (v - v^j) dx - (u - u^j) dy - \frac{S}{a^2} B_1^j dx \right] = 0 \quad (3.100)$$

Equation (3.100) may be written in the more convenient form

$$d\theta \mp \cot \alpha \frac{dP}{\rho q^2} \mp \frac{\delta \sin \theta \sin \alpha dx}{y \cos(\theta \mp \alpha)} \pm \frac{dx}{\rho q^2 \cos(\theta \mp \alpha)} \sum_{j=1}^{NP} \rho^j A^j \left[ \pm (v - v^j) \cos(\theta \mp \alpha) \right. \\ \left. \mp (u - u^j) \sin(\theta \mp \alpha) + \frac{B_1^j}{q \sin \alpha} \right] \mp \frac{\frac{dx}{C_p/R - 1} \sum_{i=1}^{NG} \mu_i \dot{X}_1}{q^3 \sin \alpha \cos(\theta \mp \alpha)} = 0 \quad (3.136)$$

The particle streamline direction,  $\theta^j$ , is given by

$$\frac{dy}{dx} = \frac{v^j}{u^j} = \tan \theta^j \quad j = 1, NP \quad (3.42)$$

and the compatibility equations which apply along particle streamlines are:

$$u^j du^j = A^j (u - u^j) dx \quad j = 1, NP \quad (3.108)$$

$$u^j dv^j = A^j (v - v^j) dx \quad j = 1, NP \quad (3.109)$$

and

$$u^j dh^j = - \left[ \frac{2}{3} A^j C^j (T^j - T) + \frac{3\sigma}{m^j r^j} \left[ \epsilon^j (T^j)^4 - \alpha^j T^4 \right] \right] dx \quad j = 1, NP \quad (3.111)$$

### Enthalpy-Entropy-Velocity Form (For Chemical Equilibrium and/or Frozen Flow Applications)

The slope of the gas streamline,  $\theta$ , is given by

$$\frac{dy}{dx} = \tan \theta \quad (3.41)$$

and the compatibility equations which apply along gas streamlines are:

$$dH - T dS + \frac{1}{\rho} \sum_{j=1}^{NP} \rho^j A^j \left[ (u - u^j) + \frac{v}{u} (v - v^j) \right] dx = 0 \quad (3.129a)$$

$$T dS - \frac{(C_p - R)}{u \rho R} \sum_{j=1}^{NP} \rho^j A^j B_1^j dx = 0 \quad (3.128a)$$

where

$$B_1^j = \frac{1}{C_p/R - 1} \left\{ \vec{q} \cdot \Delta \vec{q}^j - \vec{q}^j \cdot \Delta \vec{q} + \frac{2}{3} C^j (T^j - T) + \frac{3\sigma}{A^j m^j r^j} \left[ \epsilon^j (T^j)^4 - \alpha^j T^4 \right] \right\} \quad (2.103)$$

$$A^j = \frac{9}{2} \left[ \frac{\nu f^j}{m^j (r^j)^2} \right] \quad (2.46)$$

and

$$C^j = \frac{k G^j}{\nu f^j} \quad (2.83)$$

The slope of the Mach lines is given by

$$\frac{dy}{dx} = \tan(\theta \mp \alpha) \quad (3.56)$$

and the compatibility equations which apply along each Mach line are:

$$d\theta \pm \frac{\cot\alpha}{q} dq \pm \frac{\sin\alpha \cos\alpha dS}{\gamma R} \mp \frac{\cot\alpha dH}{q^2} \mp \frac{\delta \sin\theta \sin\alpha dx}{y \cos(\theta \mp \alpha)} \\ \pm \frac{dx}{\rho q^2 \cos(\theta \mp \alpha)} \sum_{j=1}^{NP} \rho^j A^j \left[ \pm (v - v^j) \cos(\theta \mp \alpha) \mp (u - u^j) \sin(\theta \mp \alpha) + \frac{B_1^j}{q \sin\alpha} \right] \quad (3.135a)$$

The independent variables are  $x, y, q, \theta, P, \rho, u^j, v^j, h^j$  and  $\rho^j$  in the Pressure-Density-Velocity form, and  $x, y, q, \theta, H, S, u^j, v^j, h^j$  and  $\rho^j$  in the Enthalpy-Entropy-Velocity form.  $6+4NP$  unknowns are required to completely define the gas-particle flow at a given location in the flow field. The above equations provide only  $6+3NP$  compatibility relations. This is because a compatibility relation does not exist for the particle density and results from the assumption that the particles do not contribute to the pressure acting on a control volume in the gas-particle system. As originally shown by Kliegel, (Ref. 3), this can be resolved by utilizing the following incompressible flow relation to obtain the particle density.

$$dm^j = (2\pi)^\delta \rho^j \left\{ u^j (y^j)^\delta dy^j - v^j (y^j)^\delta dx^j \right\} \quad (3.137)$$

and

$\delta$  takes on the values

$$\delta = \begin{matrix} 0 & \text{for two-dimensional flow} \\ 1 & \text{for axisymmetric flow} \end{matrix}$$

### 3.3 FINITE DIFFERENCE SOLUTION OF THE COMPATIBILITY EQUATIONS

To solve the system of compatibility equations it is first necessary to write these equations in finite difference form. Notice that the equations describing the particle properties were derived for a discrete particle. To account for more than one particle size or species,  $n$  discrete particles are allowed in the numerical solution. The streamline of each additional particle

is an additional characteristic curve; and the solution is obtained by applying the particle compatibility equations independently for each particle along its streamline.

The equations are cast in a form suitable for the calculation of an interior point in the flow field. For special points, such as along the nozzle axis, the nozzle wall, the limiting particle streamline and a free boundary, certain boundary conditions in the flow field are known which allow some of the equations to be discarded. The characteristic net for an interior point is illustrated by Fig. 3-2.

The difference equations for the gas-particle system of compatibility equations in coefficient form is given below.

The slope of the gas streamline,  $\theta$ , is given by

$$\frac{\Delta y}{\Delta x}_{5-3} = \tan \bar{\theta}_{5,3} \quad (3.138)$$

and the compatibility equations which apply along gas streamlines are

Pressure-Density-Velocity Form for Finite Rate Chemistry

$$\bar{q}_{5,3} \Delta q_{5-3} + \frac{\Delta P_{5-3}}{\bar{\rho}_{5,3}} + C_{3p} = 0 \quad (3.139)$$

$$\Delta P_{5-3} - \bar{a}_{5,3}^2 \Delta \rho_{5-3} + C_{2p} = 0 \quad (3.140)$$

and,

$$\bar{\rho}_{5,3} \bar{u}_{5,3} \Delta X_{i,5-3} - \bar{w}_{i,5,3} \Delta x_{5-3} = 0 \quad i = 1, NG ; \quad (3.141)$$

where the barred values are averages over the step length, and the coefficients are defined as follows

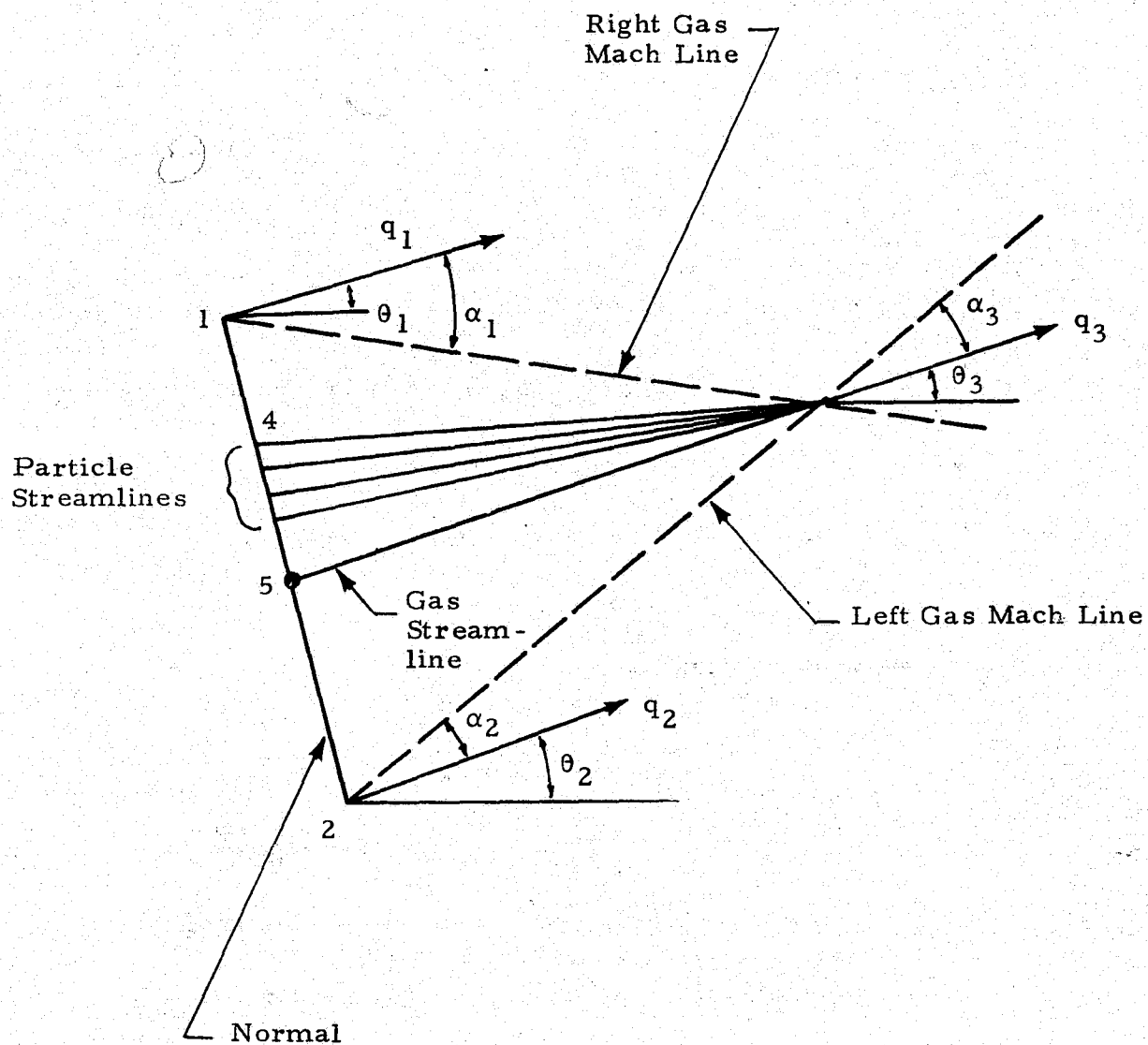


Fig. 3-2 - Characteristic Net for an Interior Point



$$C_{3p} = \frac{1}{\bar{\rho}_{5,3}} \sum_{j=1}^{NP} \bar{\rho}_{5,3}^j \bar{A}_{5,3}^j \left[ (\bar{u}_{5,3} - \bar{u}_{5,3}^j) \cos \bar{\theta}_{5,3} + (\bar{v}_{5,3} - \bar{v}_{5,3}^j) \sin \bar{\theta}_{5,3} \right] \Delta L_{5-3}$$

and

$$C_{2p} = \left\{ \left[ \frac{\bar{C}_{p5,3}}{\bar{R}_{5,3}} - 1 \right] \sum_{i=1}^{NG} \bar{\mu}_{i5,3} \bar{w}_{i5,3} - \sum_{j=1}^{NP} \bar{\rho}_{5,3}^j \bar{A}_{5,3}^j \bar{B}_{1,3}^j \right\} \frac{\Delta L_{5-3}}{\bar{q}_{5,3}}$$

$\Delta L$  is the step size along the streamline.

Enthalpy-Entropy-Velocity Form for Equilibrium and/or Frozen Chemistry

$$\Delta H_{5-3} = \bar{T}_{5,3} \Delta S_{5-3} - C_{3p} \quad (3.142)$$

and

$$\Delta S_{5-3} = C_2 \quad (3.143)$$

where

$$C_2 = \left( \frac{\bar{C}_{p5,3}}{\bar{R}_{5,3}} - 1 \right) \left[ \sum_{j=1}^{NP} \bar{\rho}_{5,3}^j \bar{A}_{5,3}^j \bar{B}_{1,3}^j \right] \frac{\Delta L_{5-3}}{\bar{\rho}_{5,3} \bar{T}_{5,3} \bar{q}_{5,3}}$$

The slope of the Mach lines (left running denoted by subscript 2 and right running denoted by subscript 1) is given by

$$\frac{\Delta y}{\Delta x}_{1,2} = \tan \bar{\beta}_{1,2} \quad (3.144)$$

where

$$\bar{\beta}_{1,2} = \bar{\theta}_{1,2} + \lambda_{1,2} \bar{\alpha}_{1,2}$$

and  $\lambda$  takes on the values

$$\lambda_1 = \begin{cases} 1 & \text{for interior solution} \\ 1 & \text{for lower boundary solution} \\ 0 & \text{for upper boundary solution} \end{cases}$$

$$\lambda_2 = \begin{cases} 1 & \text{for interior solution} \\ 0 & \text{for lower boundary solution} \\ 1 & \text{for upper boundary solution.} \end{cases}$$

The compatibility equations which apply along each Mach line are:

Pressure-Density-Velocity Form

$$\Delta\theta_{1,2} \mp Q_{1,2} \Delta P_{1,2} \mp G_{1,2} \pm C1_{1,2} = 0 \quad (3.145)$$

where the coefficients are defined as follows

$$Q_{1,2} = \frac{\cos \bar{\alpha}_{1,2}}{\sin \bar{\alpha}_{1,2} \bar{q}_{1,2}^2 \bar{\rho}_{1,2}}$$

$$G_{1,2} = \frac{\delta \sin \bar{\theta}_{1,2} \sin \bar{\alpha}_{1,2} \Delta x_{1,2}}{\bar{r}_{1,2} \cos \bar{\beta}_{1,2}}$$

and

$$C1_{1,2} = \left\{ \sum_{j=1}^{NP} \bar{\rho}_{1,2}^j \bar{A}_{1,2}^j \left[ \pm (\bar{v}_{1,2} - \bar{v}_{1,2}^j) \cos \bar{\beta}_{1,2} \mp (\bar{u}_{1,2} - \bar{u}_{1,2}^j) \sin \bar{\beta}_{1,2} \right. \right. \\ \left. \left. + \frac{\bar{B}_{1,2}^j}{\bar{q}_{1,2} \sin \bar{\alpha}_{1,2}} \right] \right\} \frac{\Delta x_{1,2}}{\bar{\rho}_{1,2} \bar{q}_{1,2}^2 \cos \bar{\beta}_{1,2}} \mp \frac{\sum_{i=1}^{NG} \bar{\mu}_{1,2} \bar{X}_{i,2}}{\bar{q}_{1,2} \sin \bar{\alpha}_{1,2} \cos \bar{\beta}_{1,2}} \frac{\Delta x_{1,2}}{\bar{C}_{P,2} - 1} \frac{1}{\bar{R}_{1,2}}$$

### Enthalpy-Entropy-Velocity Form

$$\Delta \theta_{1,2} \pm Q_{1,2} \Delta q_{1,2} \pm B_{1,2} \mp Cl_{1,2} \Delta H_{1,2} \mp G_{1,2} \pm Cl_{1,2} = 0 \quad (3.146)$$

where the coefficients are defined as follows

$$Q_{1,2} = \frac{\cos \bar{\alpha}_{1,2}}{\sin \bar{\alpha}_{1,2} \bar{q}_{1,2}}$$

$$B_{1,2} = \frac{\sin \bar{\alpha}_{1,2} \cos \bar{\alpha}_{1,2} \Delta S_{1,2}}{\bar{\alpha}_{1,2} \bar{R}_{1,2}}$$

$$Cl_{1,2} = \frac{\cos \bar{\alpha}_{1,2}}{\sin \bar{\alpha}_{1,2} \bar{q}_{1,2}^2}$$

$$G_{1,2} = \frac{\delta \sin \bar{\theta}_{1,2} \sin \bar{\alpha}_{1,2} \Delta x_{1,2}}{\bar{y}_{1,2} \cos \bar{\beta}_{1,2}}$$

and

$$Cl_{1,2} = \left\{ \sum_{j=1}^{NP} \bar{\rho}_{1,2}^j \bar{A}_{1,2}^j \left[ \pm (\bar{v}_{1,2} - \bar{v}_{1,2}^j) \cos \bar{\beta}_{1,2} \mp (\bar{u}_{1,2} - \bar{u}_{1,2}^j) \sin \bar{\beta}_{1,2} \right. \right. \\ \left. \left. + \frac{\bar{B}_{1,2}^j}{\bar{q}_{1,2} \sin \bar{\alpha}_{1,2}} \right] \right\} \frac{\Delta x_{1,2}}{\bar{\rho}_{1,2} \bar{q}_{1,2}^2 \cos \bar{\beta}_{1,2}}$$

Finally, the particle streamline direction,  $\theta^j$ , is given by

$$\frac{\Delta y^j}{\Delta x^j} = \tan \bar{\theta}_{4,3}^j \quad j = 1, NP \quad (3.147)$$

and the compatibility equations which apply along particle streamlines are:

$$\Delta u_{4-3}^j = \overline{C}_2^j_{4,3} \Delta x_{4-3}^j \quad j = 1, NP \quad (3.148)$$

$$\Delta v_{4-3}^j = \overline{C}_3^j_{4,3} \Delta x_{4-3}^j \quad j = 1, NP \quad (3.149)$$

and

$$\Delta h_{4-3}^j = \overline{C}_4^j_{4,3} \Delta x_{4-3}^j \quad j = 1, NP \quad (3.150)$$

where the coefficients are defined as follows

$$C_2^j_{4,3} = A^j_{4,3} \frac{(u_{4,3} - u_{4,3}^j)}{u_{4,3}^j} \quad (3.150a)$$

$$C_3^j_{4,3} = A^j_{4,3} \frac{(v_{4,3} - v_{4,3}^j)}{u_{4,3}^j}$$

and

$$C_4^j_{4,3} = \frac{- \left\{ \frac{2}{3} A^j_{4,3} C^j_{4,3} (T^j_{4,3} - T_{4,3}) + \frac{3\sigma_{4,3}}{m^j_{4,3} r^j_{4,3}} \left[ C^j_{4,3} (T^j_{4,3})^4 - \alpha^j_{4,3} T_{4,3}^4 \right] \right\}}{u_{4,3}^j}$$

The difference form of Kliegel's incompressible flow relation is

$$\Delta m^j_{1,2} = (2\pi)^\delta \overline{\rho}^j_{1,2} \left[ \overline{u}^j_{1,2} (\overline{y}^j_{1,2})^\delta \Delta y^j_{1,2} - \overline{v}^j_{1,2} (\overline{y}^j_{1,2})^\delta \Delta x^j_{1,2} \right]. \quad (3.151)$$

An iterative solution is employed to determine the properties of the flow at the new point. For the first pass of the solution the barred values are approximated by the conditions at the known points; e.g.,

$$\overline{\theta}_1 = \theta_1$$

After the appropriate set of equations has been solved a new estimate of the barred values is made, e.g.,

$$\overline{\theta}_1 = \frac{\theta_1 + \theta_3}{2}$$

The iterative process is continued until the desired convergence is reached.

The mechanics of the numerical solution for a typical point are quite involved. Section 6 presents the mechanics of the numerical solution for the different types of points encountered in a typical gas-particle flow problem.

### 3.4 DEVELOPMENT OF THE PARTICLE DENSITY RELATIONSHIPS

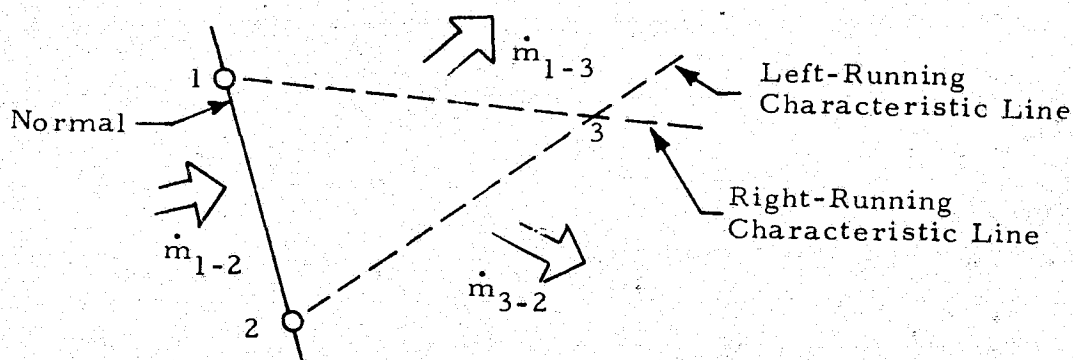
Kliegel's incompressible flow relation

$$dm^j = (2\pi)^\delta \rho^j \left\{ u^j (y^j)^\delta dy^j - v^j (y^j)^\delta dx^j \right\} \quad (3.137)$$

will now be used to develop the particle density relationships required to solve for the particle density at each of the different types of points encountered in a typical gas-particle flow problem. The particle density relationships will be derived for an interior point, a lower boundary point, a particle limiting streamline point and an upper boundary point. The particle density relationships for the remaining type points (such as a shock point) will not be developed because each is subject to the same boundary conditions as one of the above four cases and therefore yield the same results.

#### 3.4.1 Interior Point Solution

Consider the characteristic net for an interior point



Applying the principle of conservation of mass to the above system, we may write

$$\dot{m}_{1-2} = \dot{m}_{1-3} + \dot{m}_{3-2} \quad (3.152)$$

Each term in the above equation may be solved for by applying Eq. (3.137) between the proper points.

Points 1-3:

$$\dot{m}_{1-3} = (2\pi)\delta \left[ \frac{(\rho_1 u_1 + \rho_3 u_3)}{2} \frac{(y_1 y_1^\delta - y_3 y_3^\delta)}{2^\delta} - \frac{(\rho_1 v_1 + \rho_3 v_3)}{2} \frac{(y_1^\delta + y_3^\delta)}{2^\delta} (x_1 - x_3) \right] \quad (3.153)$$

Points 3-2:

$$\dot{m}_{3-2} = (2\pi)\delta \left[ \frac{(\rho_2 u_2 + \rho_3 u_3)}{2} \frac{(y_3 y_3^\delta - y_2 y_2^\delta)}{2^\delta} - \frac{(\rho_2 v_2 + \rho_3 v_3)}{2} \frac{(y_2^\delta + y_3^\delta)}{2^\delta} (x_3 - x_2) \right] \quad (3.154)$$

and points 1-2:

$$\dot{m}_{1-2} = (2\pi)\delta \left[ \frac{(\rho_1 u_1 + \rho_2 u_2)}{2} \frac{(y_1 y_1^\delta - y_2 y_2^\delta)}{2^\delta} - \frac{(\rho_1 v_1 + \rho_2 v_2)}{2} \frac{(y_1^\delta + y_2^\delta)}{2^\delta} (x_1 - x_2) \right] \quad (3.155)$$

The particle density relationship at point 3 may now be determined by substituting Eqs. (3.153), (3.154) and (3.155) into Eq. (3.152) and solving for  $\rho_3$ . Equation (3.152) becomes:

$$\begin{aligned} & (\rho_1 u_1 + \rho_2 u_2) (y_1 y_1^\delta - y_2 y_2^\delta) - (\rho_1 v_1 + \rho_2 v_2) (y_1^\delta + y_2^\delta) (x_1 - x_2) \\ &= (\rho_1 u_1 + \rho_3 u_3) (y_1 y_1^\delta - y_3 y_3^\delta) - (\rho_1 v_1 + \rho_3 v_3) (y_1^\delta + y_3^\delta) (x_1 - x_3) \\ &+ (\rho_2 u_2 + \rho_3 u_3) (y_3 y_3^\delta - y_2 y_2^\delta) - (\rho_2 v_2 + \rho_3 v_3) (y_2^\delta + y_3^\delta) (x_3 - x_2) \quad (3.156) \end{aligned}$$

To solve for  $\rho_3$ :

expand Eq. (3.156)

$$\begin{aligned} & \rho_1 u_1 (y_1 y_1^\delta - y_2 y_2^\delta) - \rho_1 v_1 (y_1^\delta + y_2^\delta) (x_1 - x_2) + \rho_2 u_2 (y_1 y_1^\delta - y_2 y_2^\delta) \\ & - \rho_2 v_2 (y_1^\delta + y_2^\delta) (x_1 - x_2) = \rho_3 u_3 (y_1 y_1^\delta - y_3 y_3^\delta) \\ & - \rho_1 v_1 (y_1^\delta + y_3^\delta) (x_1 - x_3) + \rho_3 u_3 (y_3 y_3^\delta - y_2 y_2^\delta) + \rho_1 u_1 (y_1 y_1^\delta - y_3 y_3^\delta) \\ & + \rho_2 u_2 (y_3 y_3^\delta - y_2 y_2^\delta) - \rho_2 v_2 (y_2^\delta + y_3^\delta) (x_3 - x_2) - \rho_3 v_3 (y_1^\delta + y_3^\delta) (x_1 - x_3) \\ & - \rho_3 v_3 (y_2^\delta + y_3^\delta) (x_3 - x_2) ; \end{aligned}$$

combine terms to form coefficients of  $\rho_1, \rho_2$  and  $\rho_3$

$$\begin{aligned} & \rho_1 \left[ u_1 (y_1 y_1^\delta - y_2 y_2^\delta) - v_1 (y_1^\delta + y_2^\delta) (x_1 - x_2) \right] + \rho_2 \left[ u_2 (y_1 y_1^\delta - y_2 y_2^\delta) \right. \\ & \left. - v_2 (y_1^\delta + y_2^\delta) (x_1 - x_2) \right] = \rho_3 \left[ u_3 (y_1 y_1^\delta - y_3 y_3^\delta + y_3 y_3^\delta - y_2 y_2^\delta) \right. \\ & \left. - v_3 (y_1^\delta + y_3^\delta) (x_1 - x_3) - v_3 (y_2^\delta + y_3^\delta) (x_3 - x_2) \right] \\ & + \rho_1 \left[ u_1 (y_1 y_1^\delta - y_3 y_3^\delta) - v_1 (y_1^\delta + y_3^\delta) (x_1 - x_3) \right] + \rho_2 \left[ u_2 (y_3 y_3^\delta - y_2 y_2^\delta) \right. \\ & \left. - v_2 (y_2^\delta + y_3^\delta) (x_3 - x_2) \right] ; \end{aligned}$$

isolate  $\rho_3$  and its coefficient

$$\begin{aligned} & \rho_3 \left\{ u_3 (y_1 y_1^\delta - y_2 y_2^\delta) - v_3 \left[ (y_1^\delta + y_3^\delta) (x_1 - x_3) + (y_2^\delta + y_3^\delta) (x_3 - x_2) \right] \right\} \\ & = \rho_1 \left[ u_1 (y_1 y_1^\delta - y_2 y_2^\delta) - v_1 (y_1^\delta + y_2^\delta) (x_1 - x_2) - u_1 (y_1 y_1^\delta - y_3 y_3^\delta) \right. \end{aligned}$$

$$+ v_1(y_1^\delta + y_3^\delta)(x_1 - x_3)] + \rho_2[u_2(y_1 y_1^\delta - y_2 y_2^\delta) - v(y_1^\delta + y_2^\delta)(x_1 - x_2) - u_2(y_3 y_3^\delta - y_2 y_2^\delta) + v_2(y_2^\delta + y_3^\delta)(x_3 - x_2)] ;$$

combine terms on the right hand side of the equal sign

$$= \rho_1 \left\{ u_1(y_3 y_3^\delta - y_2 y_2^\delta) - v_1[(y_1^\delta + y_2^\delta)(x_1 - x_2) - (y_1^\delta + y_3^\delta)(x_1 - x_3)] \right\} + \rho_2 \left\{ u_2(y_1 y_1^\delta - y_3 y_3^\delta) - v_2[(y_1^\delta + y_2^\delta)(x_1 - x_2) - (y_2^\delta + y_3^\delta)(x_3 - x_2)] \right\} ;$$

and finally solving for  $\rho_3$

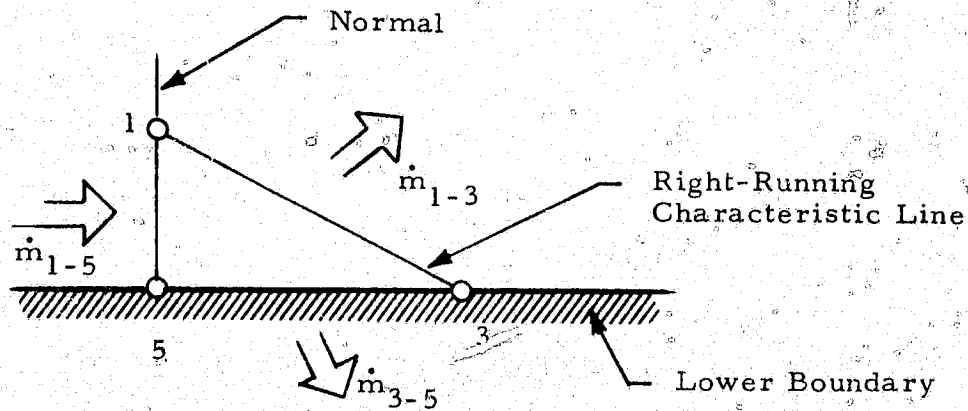
$$\rho_3 = \frac{1}{u_3(y_1 y_1^\delta - y_2 y_2^\delta) - v_3[(y_1^\delta + y_3^\delta)(x_1 - x_3) + (y_2^\delta + y_3^\delta)(x_3 - x_2)]} \left[ \rho_1 \left\{ u_1(y_3 y_3^\delta - y_2 y_2^\delta) - v_1[(y_1^\delta + y_2^\delta)(x_1 - x_2) - (y_1^\delta + y_3^\delta)(x_1 - x_3)] \right\} + \rho_2 \left\{ u_2(y_1 y_1^\delta - y_3 y_3^\delta) - v_2[(y_1^\delta + y_2^\delta)(x_1 - x_2) - (y_2^\delta + y_3^\delta)(x_3 - x_2)] \right\} \right] . \quad (3.157)$$

#### 3.4.2 Lower Boundary Point Solution

The lower wall point and free boundary point are the two types of lower boundary points encountered in a typical gas-particle flow problem. To derive a relationship for the particle density at a lower boundary point it will be assumed that: (1) the lower boundary runs parallel to the x-axis (for two-phase only), and (2) the particle streamlines run parallel to the lower boundary when in the vicinity of the lower boundary. These assumptions are valid in nearly all of the flow problems encountered.



Consider the characteristic net for a point on a lower boundary



Applying the principle of conservation of mass to the above system, we may write

$$\dot{m}_{1-5} = \dot{m}_{1-3} + \dot{m}_{3-5} \quad (3.158)$$

Furthermore, from the above assumptions  $\dot{m}_{3-5}$ ,  $v_5^j$  and  $v_3^j$  may be set equal to zero.

Equation (3.158) then becomes

$$\dot{m}_{1-5} = \dot{m}_{1-3} \quad (3.159)$$

Each term in the above equation may be solved for by applying Eq. (3.137) between the proper points.

Points 1-5:

$$\dot{m}_{1-5} = (2\pi)^\delta \left[ \frac{(\rho_1 u_1 + \rho_5 u_5) y_1 y_1^\delta}{2} - \rho_1 v_1 y_1^\delta (x_1 - x_5) \right] \quad (3.160)$$

Points 1-3:

$$\dot{m}_{1-3} = (2\pi)^\delta \left[ \frac{(\rho_1 u_1 + \rho_3 u_3) y_1 y_1^\delta}{2} - \rho_1 v_1 y_1^\delta (x_1 - x_3) \right] \quad (3.161)$$

The particle density relationship at point 3 may now be determined by equating Eqs. (3.160) and (3.161) and solving for  $\rho_3$ ,

Equation (3.159) becomes:

$$(\rho_1 u_1 + \rho_5 u_5) y_1 y_1^\delta - 2\rho_1 v_1 y_1^\delta (x_1 - x_5) = (\rho_1 u_1 + \rho_3 u_3) y_1 y_1^\delta - 2\rho_1 v_1 y_1^\delta (x_1 - x_3) \quad (3.162)$$

To solve for  $\rho_3$ :

combine terms of Eq. (3.162) to form coefficients of  $\rho_1$ ,  $\rho_3$  and  $\rho_5$

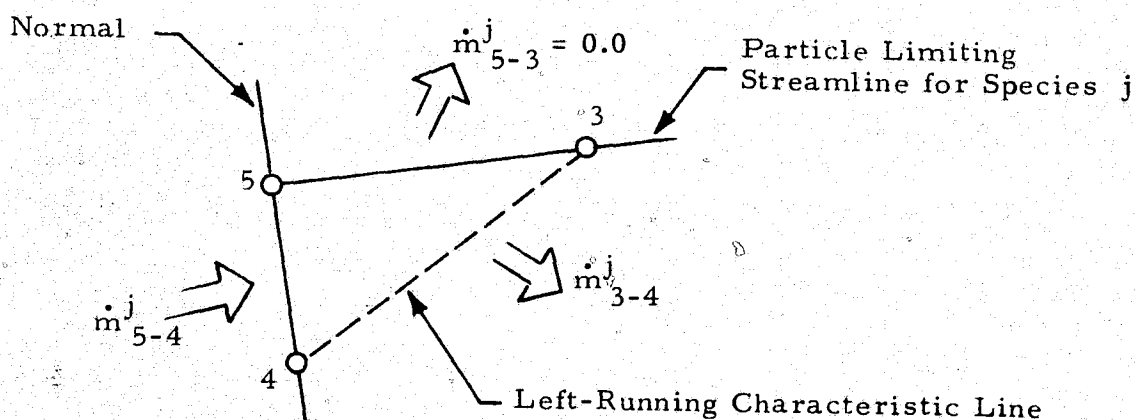
$$\rho_1 [2v_1 y_1^\delta (x_5 - x_3)] + \rho_5 u_5 y_1 y_1^\delta = \rho_3 u_3 y_1 y_1^\delta$$

and then divide through by  $u_3 y_1 y_1^\delta$

$$\rho_3 = \frac{\rho_5 u_5 - 2\rho_1 v_1 (x_3 - x_5)}{u_3} \quad (3.163)$$

### 3.4.3 Particle Limiting Streamline Point Solution

Consider the characteristic net for a point on a particle limiting streamline



Applying the principle of conservation of mass to species  $j$  in the above system, we may write

$$\dot{m}_{5-4}^j = \dot{m}_{5-3}^j + \dot{m}_{3-4}^j \quad (3.164)$$

By definition of a particle limiting streamline, the entire mass of particle species  $j$  is contained in the streamtube formed by the particle limiting streamline for species  $j$ . Therefore  $\dot{m}_{5-3}^j$  may be set equal to zero. Note that this condition applies only to particle species  $j$ ; other particle species may pass through the particle limiting streamline for species  $j$ . Equation (3.164) then becomes

$$\dot{m}_{3-4} = \dot{m}_{5-4} \quad (3.165)$$

Each term in the above equation may be solved for by applying Eq. (3.137) between the proper points.

Points 3-4:

$$\dot{m}_{3-4} = (2\pi)^\delta \left[ \frac{(\rho_3 u_3 + \rho_4 u_4)}{2} \frac{(y_3 y_3^\delta - y_4 y_4^\delta)}{2^\delta} - \frac{(\rho_3 v_3 + \rho_4 v_4)}{2} \frac{(y_3^\delta + y_4^\delta)}{2^\delta} (x_3 - x_4) \right] \quad (3.166)$$

Points 5-4:

$$\dot{m}_{5-4} = (2\pi)^\delta \left[ \frac{(\rho_5 u_5 + \rho_4 u_4)}{2} \frac{(y_5 y_5^\delta - y_4 y_4^\delta)}{2^\delta} - \frac{(\rho_5 v_5 + \rho_4 v_4)}{2} \frac{(y_5^\delta + y_4^\delta)}{2^\delta} (x_5 - x_4) \right] \quad (3.167)$$

The particle density relationship at point 3 may now be determined by equating Eqs. (3.166) and (3.167) and solving for  $\rho_3$ . Equation (3.165) becomes:

$$\begin{aligned} & (\rho_3 u_3 + \rho_4 u_4) (y_3 y_3^\delta - y_4 y_4^\delta) - (\rho_3 v_3 + \rho_4 v_4) (y_3^\delta + y_4^\delta) (x_3 - x_4) = \\ & (\rho_5 u_5 + \rho_4 u_4) (y_5 y_5^\delta - y_4 y_4^\delta) - (\rho_5 v_5 + \rho_4 v_4) (y_5^\delta + y_4^\delta) (x_5 - x_4) \end{aligned} \quad (3.168)$$

To solve for  $\rho_3$ :

expand Eq. (3.168)

$$\begin{aligned} & \rho_3 u_3 (y_3 y_3^\delta - y_4 y_4^\delta) - \rho_3 v_3 (y_3^\delta + y_4^\delta) (x_3 - x_4) + \rho_4 u_4 (y_3 y_3^\delta - y_4 y_4^\delta) \\ & - \rho_4 v_4 (y_3^\delta + y_4^\delta) (x_3 - x_4) = \rho_5 u_5 (y_5 y_5^\delta - y_4 y_4^\delta) - \rho_5 v_5 (y_5^\delta + y_4^\delta) (x_5 - x_4) \\ & + \rho_4 u_4 (y_5 y_5^\delta - y_4 y_4^\delta) - \rho_4 v_4 (y_5^\delta + y_4^\delta) (x_5 - x_4) ; \end{aligned}$$

combine terms to form coefficients of  $\rho_3$ ,  $\rho_4$  and  $\rho_5$

$$\begin{aligned} \rho_3 [u_3 (y_3 y_3^\delta - y_4 y_4^\delta) - v_3 (y_3^\delta + y_4^\delta) (x_3 - x_4)] &= \rho_5 [u_5 (y_5 y_5^\delta - y_4 y_4^\delta) \\ & - v_5 (y_5^\delta + y_4^\delta) (x_5 - x_4)] + \rho_4 \left\{ u_4 (y_5 y_5^\delta - y_3 y_3^\delta) - v_4 [(y_5^\delta + y_4^\delta) (x_5 - x_4) \right. \\ & \left. - (y_3^\delta + y_4^\delta) (x_3 - x_4)] \right\} ; \end{aligned}$$

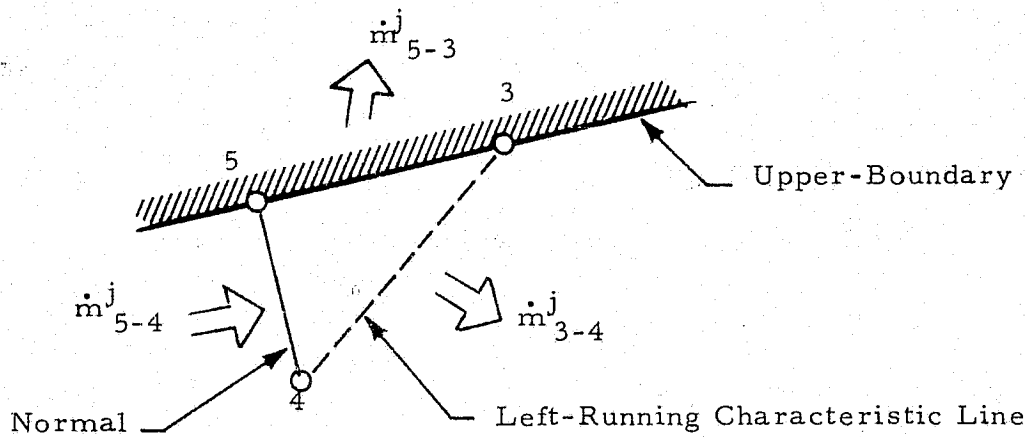
and finally solving for  $\rho_3$

$$\begin{aligned} \rho_3 &= \frac{1}{u_3 (y_3 y_3^\delta - y_4 y_4^\delta) - v_3 (y_3^\delta + y_4^\delta) (x_3 - x_4)} \left\{ \rho_5 [u_5 (y_5 y_5^\delta - y_4 y_4^\delta) \right. \\ & - v_5 (y_5^\delta + y_4^\delta) (x_5 - x_4)] + \rho_4 \left\{ u_4 (y_5 y_5^\delta - y_3 y_3^\delta) - v_4 [(y_5^\delta + y_4^\delta) (x_5 - x_4) \right. \\ & \left. - (y_3^\delta + y_4^\delta) (x_3 - x_4)] \right\} \left. \right\} . \end{aligned} \quad (3.169)$$

#### 3.4.4 Upper Boundary Point Solution

The upper wall point and free boundary point are the two types of upper boundary points encountered in a typical gas-particle flow problem. Consider

the characteristic net for a point on an upper boundary



Applying the principle of conservation of mass to species  $j$  in the above system, we may write

$$\dot{m}_{5-4}^j = \dot{m}_{5-3}^j + \dot{m}_{3-4}^j \quad (3.170)$$

Unlike the assumption made in the lower boundary solution, the particle streamlines are not restricted to run parallel to the upper boundary in the vicinity of the boundary. Therefore, the particle mass flow rate  $\dot{m}_{5-3}^j$  is not necessarily equal to zero. Physically speaking,  $\dot{m}_{5-3}^j$  may be considered to be equivalent to particles "sticking" to the upper wall or passing through the free boundary.

The particle density relationship at the upper boundary is derived by following the same mathematical procedure used in the solution for the particle density relationship at an interior point. The final result is

$$\rho_3 = \frac{1}{u_3(y_5 y_5^\delta - y_4 y_4^\delta) - v_3 \left[ (y_5^\delta + y_3^\delta)(x_5 - x_3) + (y_4^\delta + y_3^\delta)(x_3 - x_4) \right]} \left\{ \rho_5 \left[ u_5(y_3 y_3^\delta - y_4 y_4^\delta) - v_5 \left[ (y_5^\delta + y_4^\delta)(x_5 - x_4) - (y_5^\delta + y_3^\delta)(x_5 - x_3) \right] \right] + \rho_4 \left[ u_4(y_5 y_5^\delta - y_3 y_3^\delta) - v_4 \left[ (y_5^\delta + y_4^\delta)(x_5 - x_4) - (y_4^\delta + y_3^\delta)(x_3 - x_4) \right] \right] \right\} \quad (3.171)$$

## Section 4 THE OBLIQUE SHOCK SOLUTION

### 4.1 DEVELOPMENT OF OBLIQUE SHOCK RELATIONS

Figure 4-1 illustrates a streamtube passing through an oblique shock wave. This wave, which is extremely thin, will cause an almost instantaneous rise in pressure and temperature. For some distance downstream of the shock wave (in a reacting gas) a non-equilibrium zone will exist followed by a return to chemical equilibrium. The following analysis discusses the fluid flow properties in such a way that the non-equilibrium process need not be specified in order to arrive at an exact solution for the gas properties.

Consider the control surface as shown in Fig. 4-1. Applying the principle of conservation of mass, we may write

$$\rho_2 q_2 A_2 - \rho_1 q_1 A_1 = 0 \quad (4.1)$$

Similarly, conservation of momentum gives

$$\begin{aligned} & -(p_1 + \rho_1 q_1^2) A_1 \overline{\ell \theta} + \frac{p_1 A_1}{\tan \epsilon} \overline{\ell n} + (p_2 + \rho_2 q_2^2) A_2 \overline{\ell \theta'} - \frac{p_2 A_2 \overline{\ell n'}}{\tan(\epsilon - \delta)} \\ & + \int_{A_{N.E.}} p_{N.E.} \overline{dA} - \int_{A_{N.E.}} p_{N.E.} \overline{dA} = 0 \end{aligned}$$

where the subscript N. E. applies to the non-equilibrium zone. Since each streamline locally undergoes the same process the last two terms of the above

equation are equal and opposite. Therefore the above equation becomes

$$-(p_1 + \rho_1 q_1^2) A_1 \bar{l}\theta + \frac{p_1 A_1}{\tan \epsilon} \bar{l}n + (p_2 + \rho_2 q_2^2) A_2 \bar{l}\theta' - \frac{p_2 A_2 \bar{l}n'}{\tan(\epsilon - \delta)} = 0 \quad (4.2)$$

To transform Eq. (4.2) to the ns, ts coordinate system,

$$\begin{aligned} \text{Let } \bar{l}\theta &= \sin \epsilon \bar{n}s + \cos \epsilon \bar{t}s \\ \bar{l}n &= -\cos \epsilon \bar{n}s + \sin \epsilon \bar{t}s \end{aligned} \quad (4.3)$$

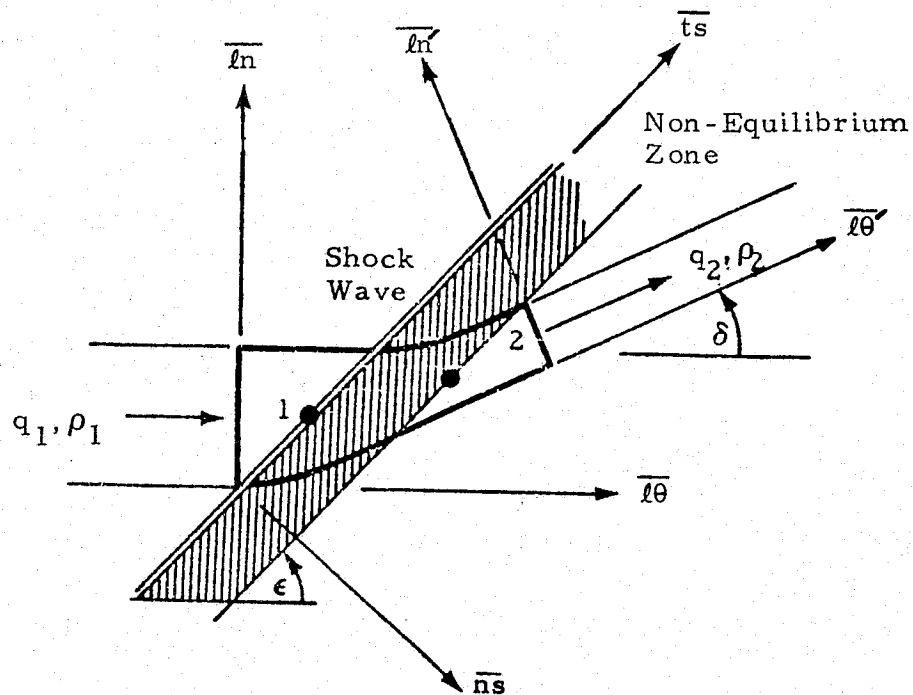


Fig. 4-1 - Nomenclature for Oblique-Shock Analysis

and

$$\begin{aligned}\overline{tn'} &= \sin(\epsilon - \delta) \overline{ns} + \cos(\epsilon - \delta) \overline{ts} \\ \overline{ln'} &= -\cos(\epsilon - \delta) \overline{ns} + \sin(\epsilon - \delta) \overline{ts} \quad .\end{aligned}\quad (4.4)$$

So that, after substitution of Eqs.(4.3) and (4.4) and setting each component to zero, Eq. (4.2) becomes,

$$\rho_2 q_2^2 \cos(\epsilon - \delta) A_2 - \rho_1 q_1^2 A_1 \cos \epsilon = 0 \quad (4.5)$$

and,

$$\begin{aligned}\left( p_1 + \rho_1 q_1^2 \right) A_1 \sin \epsilon + \frac{p_1 A_1}{\tan \epsilon} \cos \epsilon - \left( p_2 + \rho_2 q_2^2 \right) A_2 \sin(\epsilon - \delta) \\ - \frac{p_2 A_2 \cos(\epsilon - \delta)}{\tan(\epsilon - \delta)} = 0.\end{aligned}\quad (4.6)$$

But from geometry it can be seen that

$$\frac{A_2}{A_1} = \frac{\sin(\epsilon - \delta)}{\sin \epsilon} \quad . \quad (4.7)$$

After substitution of Eq. (4.7); Eqs. (4.1), (4.5) and (4.6) become

$$\rho_2 q_2 \sin(\epsilon - \delta) - \rho_1 q_1 \sin \epsilon = 0 \quad (4.8)$$

$$\rho_2 q_2^2 \sin(\epsilon - \delta) \cos(\epsilon - \delta) - \rho_1 q_1^2 \sin \epsilon \cos \epsilon = 0 \quad (4.9)$$



$$p_2 + \rho_2 q_2^2 \sin^2(\epsilon - \delta) - p_1 - \rho_1 q_1^2 \sin^2 \epsilon = 0. \quad (4.10)$$

The above set of relations contains  $\epsilon$ ,  $\delta$ ,  $p_2$ ,  $\rho_2$ ,  $q_2$  as unknown quantities, but

$$p_2 = p(s_2, q_2) ; \rho_2 = \rho(s_2, q_2). \quad (4.11)$$

Therefore, if one variable, say  $\epsilon$ , is taken as an independent parameter the remaining unknowns ( $\delta, q_2, s_2$ ) may be found by an iterative solution. These equations are, of course, formally the same as the ideal gas solution. The difference lies only in the variation of pressure etc., with entropy and velocity. It is impossible to determine the location of the new equilibrium shock point location without a detailed description of the non-equilibrium reaction process. It will be assumed therefore that this zone is thin and that no significant errors are introduced by letting the downstream physical location lie on the upstream location.

It has been pointed out previously that the characteristic equation derivation was based on neglecting transport properties and as such is necessarily restricted to continuous regions only. The oblique shock wave relations derived here then are patching lines between the continuous regions.

## 4.2 ITERATIVE SOLUTION OF THE OBLIQUE SHOCK RELATIONS

Rearranging Eq. (4.8) yields

$$\sin(\epsilon - \delta) = \frac{\rho_1 q_1 \sin \epsilon}{\rho_2 q_2} ;$$

while squaring both sides of Eq. (4.9) and substituting the above relation yields, after simplification

$$q_2 - q_1 \left\{ \left( \frac{\rho_1}{\rho_2} \right)^2 \sin^2 \epsilon + \cos^2 \epsilon \right\}^{\frac{1}{2}} = 0; \quad (4.12)$$

and Eq. (4.10) becomes

$$p_2 + \rho_1 q_1^2 \sin^2 \epsilon \left\{ \frac{\rho_1}{\rho_2} - 1 \right\} - p_1 = 0. \quad (4.13)$$

In functional form Eq. (4.12) and (4.13) are simply

$$\begin{aligned} G_1(s_2, q_2) &= 0 \\ G_2(s_2, q_2) &= 0. \end{aligned} \quad (4.14)$$

Furthermore,

$$\begin{aligned} dG_1 &= \frac{\partial G_1}{\partial q_2} dq_2 + \frac{\partial G_1}{\partial s_2} ds_2 \\ dG_2 &= \frac{\partial G_2}{\partial q_2} dq_2 + \frac{\partial G_2}{\partial s_2} ds_2. \end{aligned} \quad (4.15)$$

The partial derivatives may be written directly from Eqs. (4.12) and (4.13)

$$\frac{\partial G_1}{\partial q_2} = 1 + q_1 \left( \frac{q_1}{q_2} \right) \left( \frac{\rho_1}{\rho_2} \right)^2 \sin^2 \epsilon \frac{\partial}{\partial q_2} (\ln \rho_2)$$

$$\frac{\partial G_1}{\partial s_2} = q_1 \left( \frac{q_1}{q_2} \right) \left( \frac{\rho_1}{\rho_2} \right)^2 \sin^2 \epsilon \frac{\partial}{\partial s_2} (\ln \rho_2)$$

$$\frac{\partial G_2}{\partial q_2} = p_2 \frac{\partial}{\partial q_2} (\ln p_2) - \left( \frac{\rho_1}{\rho_2} \right)^2 q_1^2 \sin^2 \epsilon \rho_2 \frac{\partial}{\partial q_2} (\ln \rho_2) \quad (4.16)$$

$$\frac{\partial G_2}{\partial s_2} = p_2 \frac{\partial}{\partial s_2} (\ln p_2) - \left( \frac{\rho_1}{\rho_2} \right)^2 \sin^2 \epsilon \rho_2 \frac{\partial}{\partial s_2} (\ln \rho_2) .$$

Rather than calculate the partial derivatives numerically by perturbing the functions  $\ln p_2$  and  $\ln \rho_2$ , approximate values for these derivatives will be found by assuming that locally the gas behaves ideally, that is to say

$$\frac{\partial R_2}{\partial s_2} = \frac{\partial R_2}{\partial q_2} = \frac{\partial T_{o2}}{\partial s_2} = \frac{\partial T_{o2}}{\partial q_2} = \frac{\partial p_{o2}}{\partial s_2} = \frac{\partial p_{o2}}{\partial q_2} = \frac{\partial \gamma_2}{\partial s_2} = \frac{\partial \gamma_2}{\partial q_2} = 0 ;$$

so that

$$\frac{\partial (\ln p_2)}{\partial s_2} \approx \frac{\partial (\ln \rho_2)}{\partial s_2} \approx -\frac{1}{R_2} \quad (4.17)$$

and

$$\frac{\partial (\ln p_2)}{\partial q_2} \approx \gamma_2 \frac{\partial}{\partial q_2} (\ln \rho_2) \approx -\frac{q_2 \rho_2}{p_2} . \quad (4.18)$$

Writing Eq. (4.15) in finite difference form:

$$G_1^{(n+1)} - G_1^{(n)} = \frac{\partial G_1^{(n)}}{\partial q_2} \left( q_2^{(n+1)} - q_2^{(n)} \right) + \frac{\partial G_1^{(n)}}{\partial s_2} \left( s_2^{(n+1)} - s_2^{(n)} \right)$$

and

$$G_2^{(n+1)} - G_2^{(n)} = \frac{\partial G_2^{(n)}}{\partial q_2} \left( q_2^{(n+1)} - q_2^{(n)} \right) + \frac{\partial G_2^{(n)}}{\partial s_2} \left( s_2^{(n+1)} - s_2^{(n)} \right) .$$

Since the root  $G_1 = G_2 = 0$  is desired,  $G_1^{(n+1)}, G_2^{(n+1)}$  are set to zero, resulting in

$$s^{(n+1)} = s^{(n)} + \left( G_2^{(n)} \frac{\partial G_1^{(n)}}{\partial q_2} - G_1^{(n)} \frac{\partial G_2^{(n)}}{\partial q_2} \right) / \left( \frac{\partial G_1^{(n)}}{\partial s_2} \frac{\partial G_2^{(n)}}{\partial q_2} - \frac{\partial G_2^{(n)}}{\partial s_2} \frac{\partial G_1^{(n)}}{\partial q_2} \right) \quad (4.19)$$

and

$$q^{(n+1)} = q^{(n)} - \left\{ G_1^{(n)} + \frac{\partial G_1^{(n)}}{\partial s_2} \left( s^{(n+1)} - s^{(n)} \right) \right\} / \frac{\partial G_1^{(n)}}{\partial q_2} \quad (4.20)$$

The iterative solution using Eqs. (4.19) and (4.20) is continued until the desired convergence of  $G_1$  and  $G_2$  is reached. The solution is completed by

$$\delta = \epsilon - \sin^{-1} \left\{ \frac{\rho_1 q_1}{\rho_2 q_2} \sin \epsilon \right\} \quad (4.21)$$

The first guess to start the solution is an ideal gas solution to the set of equations. If it is indeed an ideal gas under analysis the first guess is

$$\delta = \epsilon - \tan^{-1} \left\{ \tan \epsilon \left( \frac{1}{M_1^2 \sin^2 \epsilon} + \frac{\gamma_1 - 1}{2} \right) \frac{2}{\gamma_1 + 1} \right\}, \quad q_2 = q_1 \frac{\cos \epsilon}{\cos(\epsilon - \delta)}$$

and

$$s_2 = s_1 + \frac{R_1}{\gamma_1 - 1} \left\{ \ln \left[ \frac{2\gamma_1 M_1^2 \sin^2 \epsilon - (\gamma_1 - 1)}{\gamma_1 + 1} \right] + \gamma_1 \ln \left[ \frac{\tan(\epsilon - \delta)}{\tan \epsilon} \right] \right\} \quad (4.22)$$

## Section 5

### EXPANSION CORNER - PRANDTL-MEYER FAN

In some cases the flow may be required to negotiate a sharp expansion turn. The problem becomes two dimensional at a sharp corner (it is impossible to conceive of an expansion corner on an axis of symmetry) and may be treated with a Prandtl-Meyer expansion.

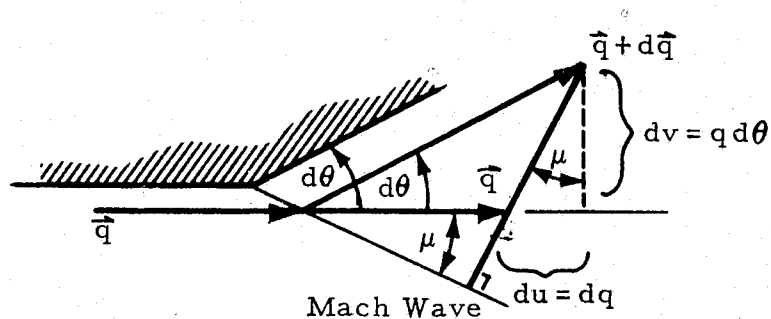


Fig. 5-1 - Nomenclature for Mach Wave Analysis

Since a Mach wave will support pressure changes only in a direction normal to itself (Ref. 11);

$$dv = q d\theta$$

$$du = dq$$

$$\frac{du}{dy} = \tan \mu = \frac{1}{\sqrt{M^2 - 1}}$$

or

$$d\theta = \sqrt{M^2 - 1} \frac{dq}{q} \quad (5.1)$$

The solution to Eq. (5.1) is a straightforward numerical integration for the case of a known final velocity (free-boundary case). If the turning angle is known, however, and the final velocity is not known, an iterative solution is necessary to determine the upper limit.

$$\int_{q_1}^{q_2} \sqrt{M^2 - 1} \frac{dq}{q} - \Delta\theta = f(q_2) = 0 \quad (5.2)$$

In the mesh construction to be discussed later a fan of rays must be generated to allow a numerical description through a large turning angle. The turning angle is subdivided into a number of small turns, each of which is integrated numerically. Corresponding to each of these small turns is a Mach wave or characteristic line emanating from the corner.

## Section 6

### NUMERICAL SOLUTION (MESH POINT CONSTRUCTION)

The calculations described previously are point or small region solutions. This section presents the mesh construction and calculation procedure required to describe the entire flow field in a typical gas-particle flow problem. The general principles adopted in the numerical solution are similar to those used in the method-of-characteristics solutions. The major difference is in the technique of mesh construction; here, the calculation proceeds along normals to the streamlines instead of along characteristic lines.

To begin the problem of describing the entire flow field all necessary boundary conditions must, of course, be supplied. In addition, a start line whose properties are completely defined must be designated.

Figure 6-1 illustrates a flow field in which there are no discontinuities and in which the mesh construction is terminated when the region of interest has been computed.

Figure 6-2a presents the mesh construction required to solve for the properties of any interior point in Fig. 6-1. The J-line is considered to be the known normal line. The J-line may be the input startline or the line just calculated. The K-line is the new normal line to be calculated. Calculation always starts from the lowest point and moves upward along the K-line. The first point on the new normal is located by extending the right-running characteristic from a properly selected point (based on desired mesh size) on the preceding known normal to intersect the lower boundary based on the flow properties of that point. The rest of the points including the last point (boundary point) of the normal are located by extending the normal to the local streamline to intersect the next streamline. (A boundary is also a



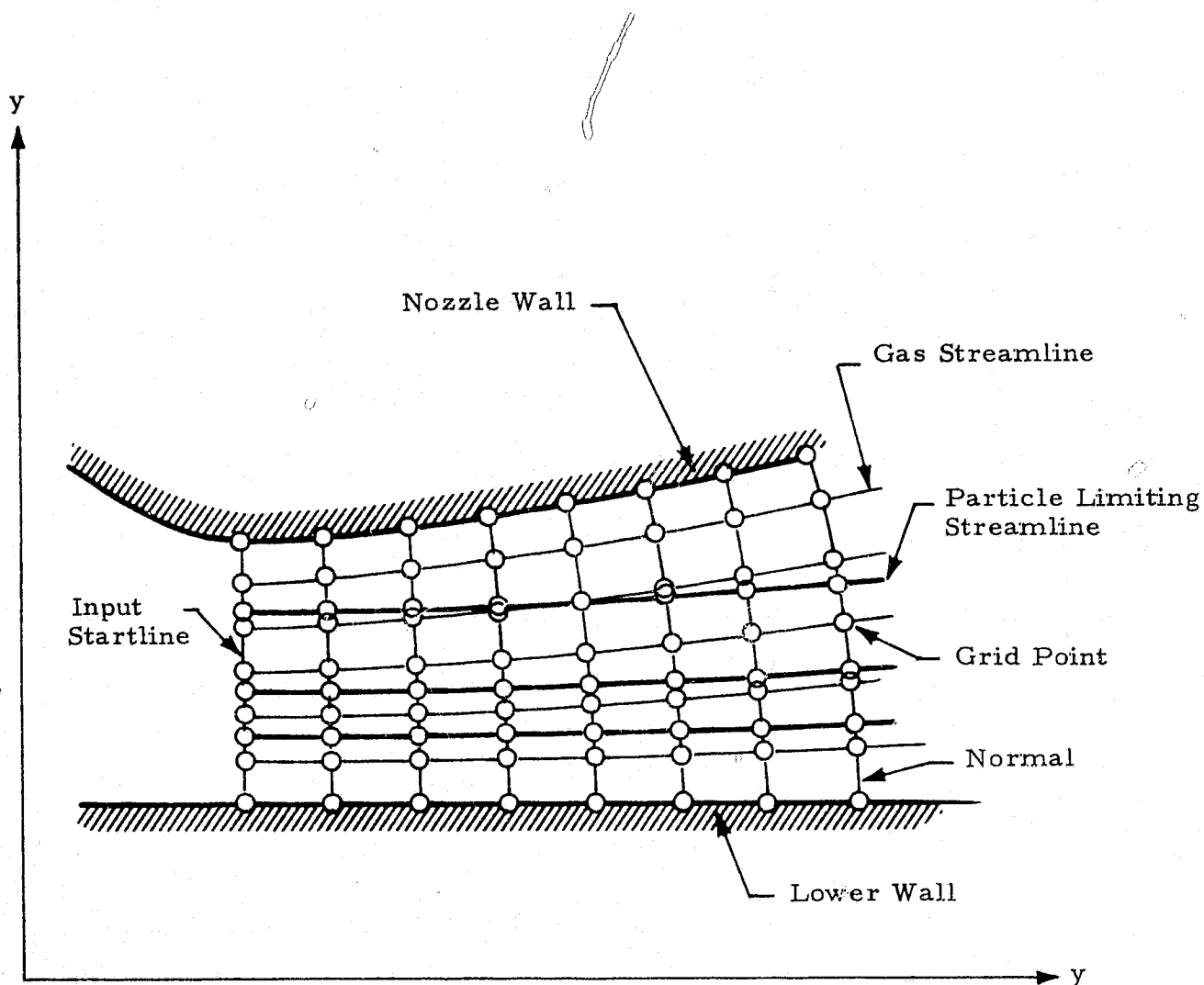


Fig.6-1 - Basic Mesh Construction for the Streamline-Normal Two-Phase Numerical Solution

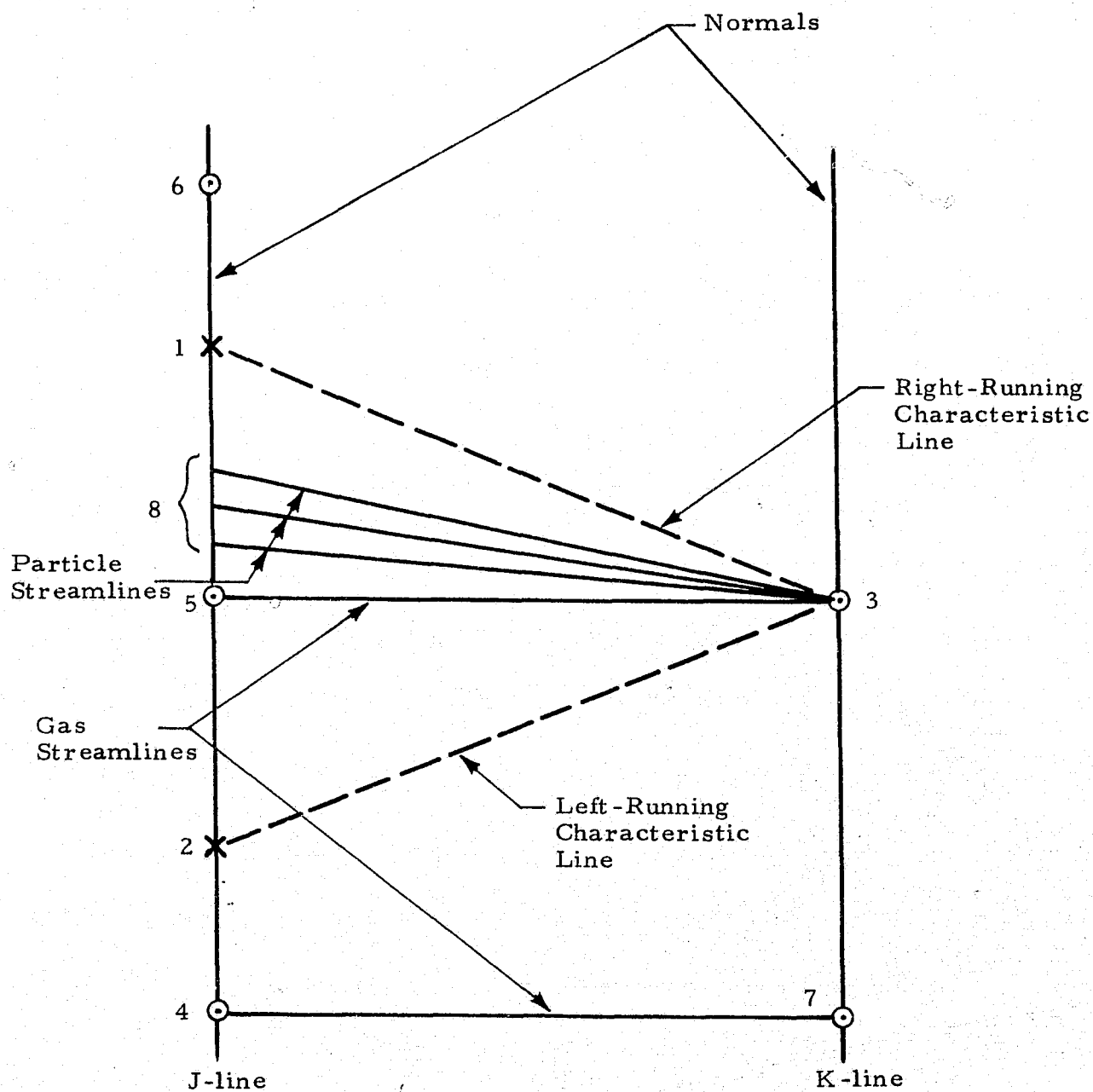


Fig. 6-2a - Mesh Construction for an Interior Point

streamline.) Once the location of the new point is known (point 3 in Fig. 6-2a), the characteristic line(s) and particle streamline(s) can be made to intersect the preceding known normal (points 1, 2 and 8 in Fig. 6-2a). The properties of points 1, 2 and 8 on the known normal can be interpolated from the known points. The properties at point 3 can then be calculated by solving the applicable compatibility equations. For an interior point, both the right-running and left-running characteristics are used to compute the velocity and the flow angle at the new point (point 3 in Fig. 6-2a). For a boundary point, either the flow angle or the velocity can be found with the boundary condition; therefore, only one physical characteristic is needed depending on the relative location of the boundary with the main flow. For shock points and Prandtl-Meyer expansion calculations, the mesh construction is appropriately modified to accommodate each individual case. The general method of computation, however, remains the same as described above.

The mesh construction and calculation procedure for an interior point solution forms the basis for the solution of the flow properties at all other types of points in the flow field. For that reason, the mesh construction and calculation procedure for an interior point will be first described in detail, then, to further aid in the understanding of the numerical solution, presented in flow chart form. Using the knowledge gained in the discussion of the interior point solution, the calculation procedure for each of the remaining cases will be described. Any important problems that may arise, or other points of interest involved in the calculational procedure will be discussed.

## 6.1 INTERIOR POINT

This section presents the details of the numerical solution for the gas and particle flow properties at an interior point of the flow field being analyzed. The solution is outlined in a step-by-step procedure and is for the case when particles are present at all points which enter into the calculation of the flow properties at the new point.

Consider again Fig. 6-2a and assume that the flow properties are known along the J-line. The J-line can be the input start line or the line just calculated. The point 7K will also be a known data point since it is the last point calculated on the K-line.

Before the calculation procedure can be started a decision must be made to use either the enthalpy-entropy-velocity form or the pressure-density-velocity form of the compatibility equations to solve for the properties at the new point. This decision is based on the type of chemistry being considered in the flow field. If finite rate chemistry is being considered, the pressure-density-velocity form of the compatibility equations is the more convenient form to be used. If equilibrium or frozen chemistry is being considered, the enthalpy-entropy-velocity form of the compatibility equations is the more convenient form to be used. In either case, the method of calculation is the same. For purposes of discussion, it will be assumed that the chemistry of the flow field being analyzed may be considered to be in chemical equilibrium.

Step 1: For the first pass of the solution, the properties at 3K are set equal to those of the base point 5J.

Step 2: The physical location  $(x_3, y_3)$  of the point 3K is found by intersecting the normal from the point 7K with the projection of the streamline from the point 5J.

Step 3: The intersection of the characteristic lines with the J-line is now found. These lines will in general fall between two known data points and an interpolation scheme is employed to find the flow properties at these points. The interpolation is of the following form:

$$P_{1,2} = P_{5,4} + F(P_{6,5} - P_{5,4})$$

where

$F$  = interpolation factor.

For the gas flow properties  $P$  assumes the following

with

$$P = q, \theta, S, H$$

$$T = f(q, S, H)$$

$$\rho = f(q, S, H)$$

$$\nu = f(T, H)$$

$$C_P = f(T, H)$$

$$Pr = f(T, H)$$

} tabulated

while for the particle properties  $P$  assumes

$$P = u^j, v^j, h^j, \rho^j$$

with

$$T_{1,2}^j = f(h^j)_{1,2} \text{ tabulated}$$

$$Re_{1,2}^j = \frac{2 r_{1,2}^j \rho_{1,2} |\Delta q^j|_{1,2}}{\nu_{1,2}}$$

$$f_{1,2} = f(Re^j)_{1,2} \text{ tabulated}$$

$$G_{1,2}^j = f(Re_{1,2}^j, Pr_{1,2}) \text{ Ref. 12}$$

$$A_{1,2}^j = \frac{9}{2} \left( \frac{f_{1,2}^j}{m_{1,2}^j (r_{1,2}^j)^2} \right) \quad (2.46)$$

$$C_{1,2}^j = \frac{G_{1,2}^j C_{p1,2}}{f_{1,2}^j Pr_{1,2}} \quad (2.83)$$

and,

$$B_{1,2}^j = \frac{1}{\frac{C_{p1,2}}{R_{1,2}} - 1} \left\{ \bar{q}_{1,2} \cdot \Delta \bar{q}_{1,2}^j - \bar{q}_{1,2}^j \cdot \Delta \bar{q}_{1,2} + \frac{2}{3} C_{1,2}^j (T_{1,2}^j - T_{1,2}) \right. \\ \left. + \frac{3\sigma_{1,2}}{A_{1,2}^j m_{1,2}^j r_{1,2}^j} \left[ \epsilon_{1,2}^j (T_{1,2}^j)^4 - \alpha_{1,2}^j T_{1,2}^4 \right] \right\} \quad (2.86c)$$

Note that the points 1, 2 and 8 are not necessarily as shown in Fig. 6-2a. Their locations can vary along the J-line depending on the mesh size and the Mach number in the vicinity of point 3. Figure 6-2b presents one example when the point  $m+2$  is an upper boundary point and the point  $1'$  is outside the flow field under consideration. Instead of the point  $1'$  the point 1 is used in the calculation. However, the properties at point 1 are not available nor can they be interpolated because point  $n+2$  is not yet known. For this particular case, the flow properties at point 1 are assumed to be identical to those of point  $m+2$  so that the properties at point 3 can be approximated. The same method is used to calculate point  $n+1$ . The boundary point  $n+2$  of the K-line can then be calculated without problem. Once the boundary point is determined the calculation goes back to point 1 and the right running characteristic

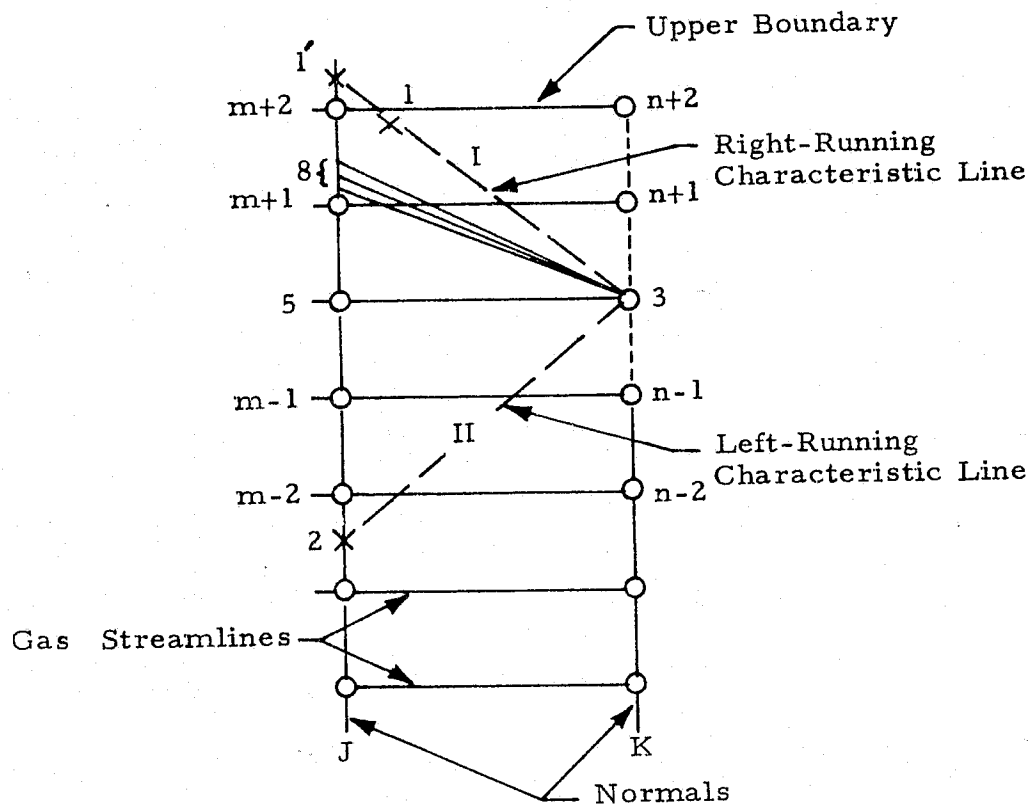


Fig. 6-2b - Mesh Construction for an Interior Point

intersection is made using the two boundary points. Point 1 is then solved for and the solution proceeds to the plume boundary.

Step 4: The particle density at point 3K is calculated using the relationship:

$$\rho_3^j = \frac{1}{u_3^j (y_1^j y_1^{j\delta} - y_2^j y_2^{j\delta}) - v_3^j \left[ (y_1^{j\delta} + y_3^{j\delta})(x_1^j - x_3^j) + (y_2^{j\delta} + y_3^{j\delta})(x_3^j - x_2^j) \right]} \left[ \rho_1^j \left\{ u_1^j (y_3^j y_3^{j\delta} - y_2^j y_2^{j\delta}) - v_1^j \left[ (y_1^{j\delta} + y_2^{j\delta})(x_1^j - x_2^j) - (y_1^{j\delta} + y_3^{j\delta})(x_1^j - x_3^j) \right] \right\} + \rho_2^j \left\{ u_2^j (y_1^j y_1^{j\delta} - y_3^j y_3^{j\delta}) - v_2^j \left[ (y_1^{j\delta} + y_2^{j\delta})(x_1^j - x_2^j) - (y_2^{j\delta} + y_3^{j\delta})(x_3^j - x_2^j) \right] \right\} \right] \quad (3.157)$$

Step 5: The particle streamline intersections with the J data line are now located (point 8 of Fig. 6-2a) using the particle streamline relationship:

$$\frac{y_3^j - y_8^j}{x_3^j - x_8^j} = \frac{v_3^j}{u_3^j} \quad (3.42)$$

Step 6: The gas and particle properties at point 8J are now found by employing an interpolation scheme as described in Step 3.

Step 7: The coefficients for the particle compatibility relations are now calculated from Eq. (3-150a).

$$C_{2,3}^j = \frac{(u_{8,3} - u_{8,3}^j)}{u_{8,3}^j} A_{8,3}^j$$

$$C_{3,3}^j = \frac{(v_{8,3} - v_{8,3}^j)}{u_{8,3}^j} A_{8,3}^j$$

$$C_{4,3}^j = \frac{- \left\{ \frac{2}{3} A_{8,3}^j C_{8,3}^j (T_{8,3}^j - T_{8,3}) + \frac{3\sigma_{8,3}}{m_{8,3}^j r_{8,3}^j} \left[ e_{8,3}^j (T_{8,3}^j)^4 - \alpha_{8,3}^j T_{8,3}^4 \right] \right\}}{u_{8,3}^j}$$

Step 8: The new particle properties at point 3K are then found by expanding the difference Eqs. (3.148), (3.149) and (3.150) to obtain

$$u_3^j = u_8^j + C_{2,3}^j \Delta x_{8-3}^j$$



$$v_3^j = v_8^j + \overline{C} 3_{8,3}^j \Delta x_{8-3}^j$$

$$h_3^j = h_8^j + \overline{C} 4_{8,3}^j \Delta x_{8-3}^j$$

with the remaining properties defined as

$$T_3^j = f(h)_3^j \quad \text{tabulated}$$

$$Re_3^j = \frac{2 r_3^j \rho_3 |\Delta q_3^j|}{\nu_3}$$

$$f_3^j = f(Re_3^j) \quad \text{tabulated}$$

$$G_3^j = f(Re_3^j, Pr_3) \quad \text{Ref. 12}$$

$$A_3^j = \frac{9}{2} \frac{f_3^j \nu_3}{m_3^j (r_3^j)^2} \quad (2.46)$$

$$C_3^j = \frac{G_3^j C_{p3}}{f_3^j Pr_3} \quad (2.83)$$

and,

$$B_{13}^j = \frac{1}{C_{p3}/R_3 - 1} \left\{ \bar{q}_3 \cdot \Delta \bar{q}_3^j - \bar{q}_3^j \cdot \Delta \bar{q}_3^j + \frac{2}{3} C_3^j (T_3^j - T_3) \right. \\ \left. + \frac{3\sigma_3}{A_3^j m_3^j r_3^j} \left[ \epsilon_3^j (T_3^j)^4 - \alpha_3^j T_3^4 \right] \right\} \quad (2.86c)$$

Step 9: The coefficients for the gas streamline compatibility relations are now calculated

$$C_2 = \left( \frac{\bar{C}_{p_{5,3}}}{\bar{R}_{5,3}} - 1 \right) \left[ \sum_{j=1}^{NP} \bar{\rho}_{5,3}^j \bar{A}_{5,3}^j \bar{B}_{1,3}^j \right] \frac{\Delta L_{5-3}}{\bar{\rho}_{5,3} \bar{T}_{5,3} \bar{q}_{5,3}}$$

and

$$C_{3_p} = \frac{1}{\bar{\rho}_{5,3}} \sum_{j=1}^{NP} \bar{\rho}_{5,3}^j \bar{A}_{5,3}^j \left[ (\bar{u}_{5,3} - \bar{u}_{5,3}^j) \cos \bar{\theta}_{5,3} + (\bar{v}_{5,3} - \bar{v}_{5,3}^j) \sin \bar{\theta}_{5,3} \right] \Delta L_{5-3} .$$

Step 10: The rate of change of entropy in the direction of the gas streamline and the entropy at the point 3K are calculated in the following manner:

$$\Delta S_{5-3} = C_2 \quad (3.143)$$

and the entropy at point 3K is

$$S_3 = S_5 + C_2 .$$

Step 11: The rate of change of enthalpy in the direction of the gas streamline and the enthalpy at the point 3K are calculated in the following manner:

$$\Delta H_{5-3} = \bar{T}_{5,3} \Delta S_{5-3} - C_{3_p} \quad (3.142)$$

and the enthalpy at point 3K is

$$H_3 = H_5 + \bar{T}_{5,3} \Delta S_{5-3} - C_{3p}$$

Step 12: The remaining gas properties to be calculated at point 3K are the velocity and flow angle. Expanding equation (3.146) in terms of both the right and left-running characteristics and solving for the velocity gives

$$q_3 = \frac{\theta_1 - \theta_2 + Q_1 q_1 + Q_2 q_2 - B_1 - B_2 + C_{I1}(H_3 - H_1) + C_{I2}(H_3 - H_2) + G_1 + G_2 - C_{I1} - C_{I2}}{Q_1 + Q_2}$$

The new flow angle is then

$$\theta_3 = \theta_2 + Q_2 q_3 - Q_2 q_2 + B_2 - C_{I2}(H_3 - H_2) - G_2 + C_{I2}$$

Step 13: The properties  $T_3, \rho_3, \nu_3, C_{p3}, Pr_3, A_3^j, C_3^j$  and  $B_3^j$  are recalculated to reflect the new gas properties. The solution is then checked to determine if the desired accuracy has been obtained. If the solution has not converged, return to step 2 and repeat the process. The flow of the numerical solution is summarized in Chart 6-1.

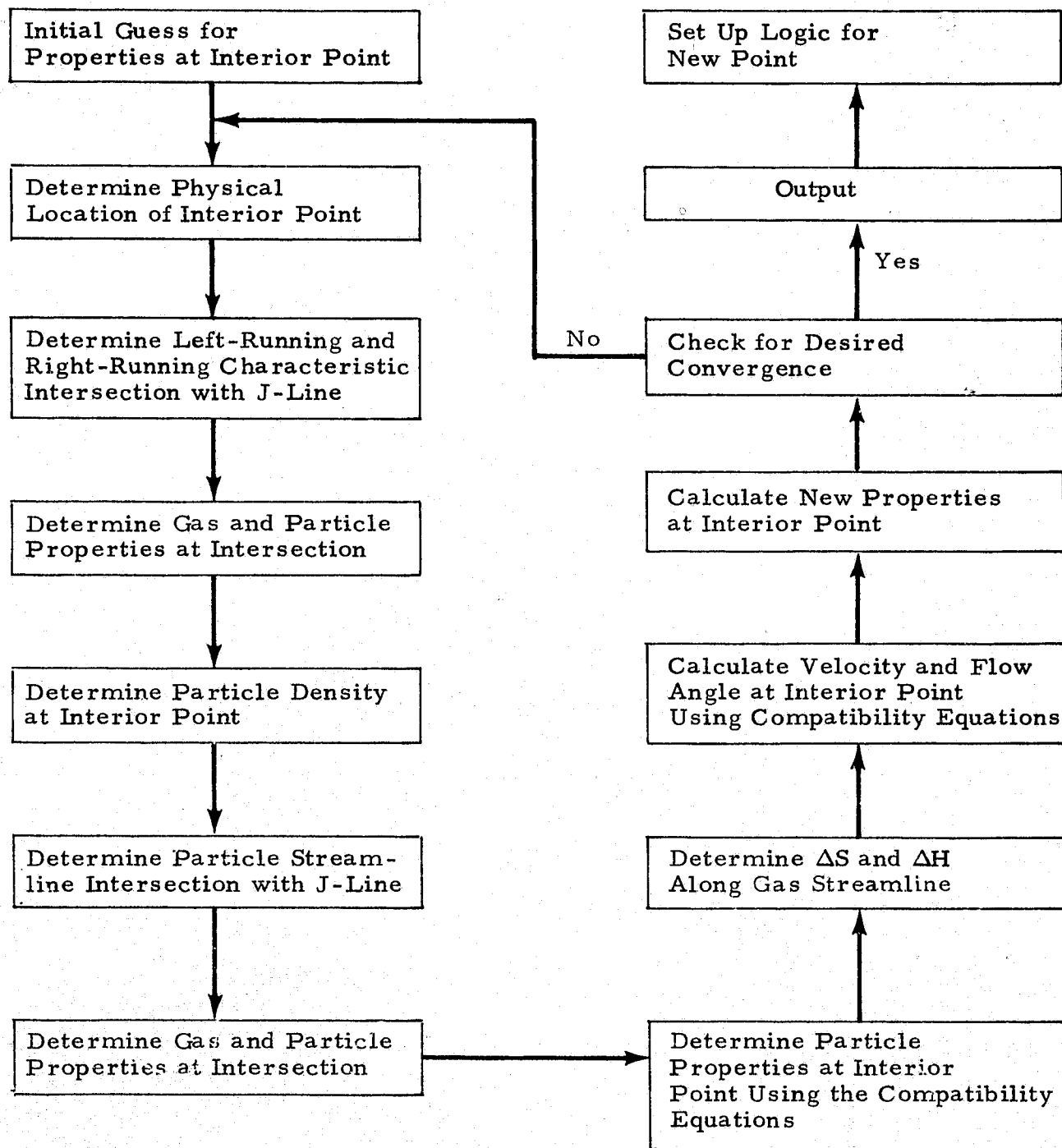


Chart 6-1 - Flow Chart of the Calculation Procedure for an Interior Point Solution

## 6.2 LOWER WALL POINT

Figure 6-3 presents the mesh construction required to solve for gas and particle properties at any lower wall point in Figure 6-1.

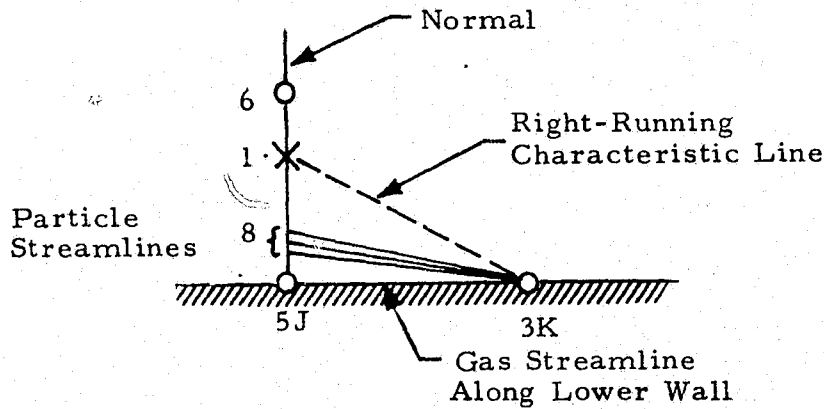


Fig. 6-3 - Mesh Construction for a Lower Wall Point

The calculational procedure followed to determine the flow properties at a lower wall point is very similar to that of an interior point solution. The lower wall point is always the first point calculated on a K-line. Before the point 3K can be located, point 1J must first be selected as a function of the desired mesh size. Once point 1J is located and its properties determined using the interpolation scheme employed for an interior point; the physical location  $(x_3, y_3)$  of point 3K is found by intersecting the right-running characteristic from point 1J and the projection of the streamline from the point 5J. The particle density at point 3K is then calculated using the relationship

$$\rho_3^j = \frac{\rho_5^j u_5^j - 2\rho_1^j v_1^j (x_3 - x_5)}{u_3^j} \quad (3.163)$$

and is based on the assumptions that: (1) the lower boundary runs parallel to the X-axis; and (2) the particle streamlines run parallel to the lower boundary when in the vicinity of the lower boundary.

Finally, since the flow angle along the lower wall is known, only the right running characteristic line is needed to calculate the velocity at the lower wall point. Solving for velocity in Eq. (3.146) gives

$$q_3 = \frac{[\theta_1 - \theta_3 + Q_1 q_1 - B_1 + C_1 I_1 (H_3 - H_1) + G_1 - C_1 I_1]}{Q_1}$$

Once the velocity is found, the calculational procedure is continued as in the case of an interior point.

### 6.3 UPPER WALL POINT

Figure 6-4 presents the mesh construction required to solve for the gas and particle properties at any upper wall point in Fig. 6-1.

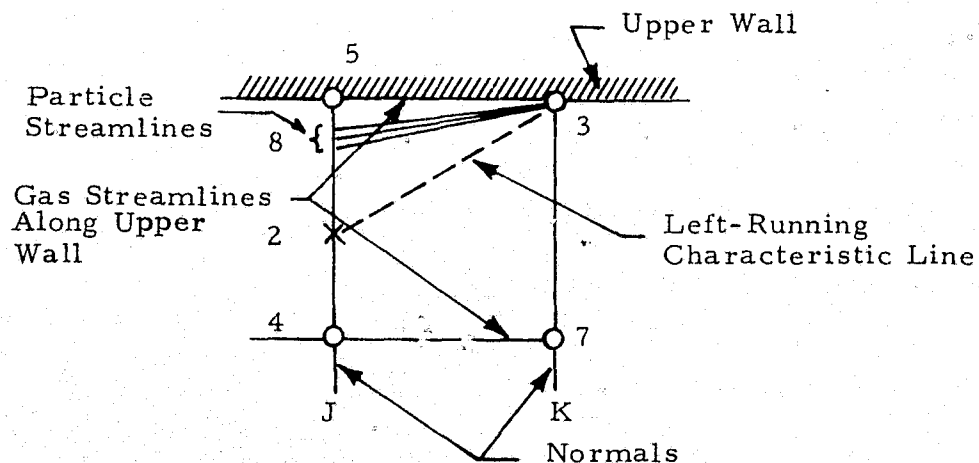


Fig. 6-4 - Mesh Construction for an Upper Wall Point

The calculational procedure followed to determine the flow properties at an upper wall point is almost identical to that of an interior point solution. The exception is that the flow angle on the upper wall is known and therefore, only the left-running characteristic line is needed to calculate the velocity at the upper boundary point. Solving for velocity in Eq. (3.146) gives

$$q_3 = \frac{\theta_3 - \theta_2 + Q_2 q_2 - B_2 + Cl_2(H_3 - H_2) + G_2 - Cl_2}{Q_2}$$

#### 6.4 FREE BOUNDARY POINT

Figure 6-5 presents the mesh construction required to solve for the gas and particle properties at a free boundary point. Note that the mesh construction is the same as that for an upper wall point.

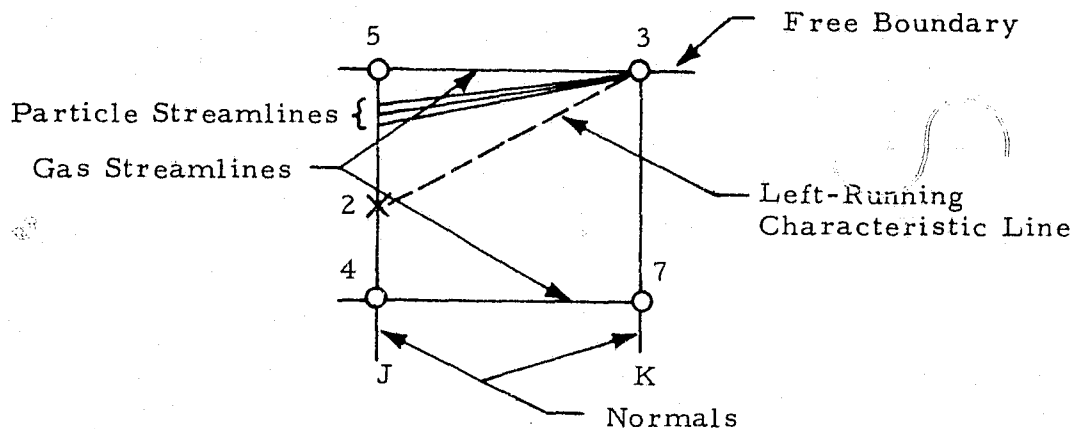


Fig. 6-5 - Mesh Construction for a Free Boundary Point

The calculational procedure followed to determine the flow properties at a free boundary point is like that of an upper wall point solution. The one difference is that the velocity and not the flow angle at the free boundary point is known. Applying Eq. (3.146) to the left-running characteristic and solving for the flow angle at the free boundary point gives

$$\theta_3 = \theta_2 + Q_2(q_3 - q_2) + B_2 - Cl_2(H_3 - H_2) - G_2 + Cl_2$$

The velocity at the free boundary, though assumed to be known, is not easily obtained. The velocity at the free boundary must be solved for iteratively using the relation

$$v_3 = \left\{ \frac{2\gamma RT}{\gamma - 1} \left[ \left( \frac{P}{P_0} \right)^{-\frac{\gamma-1}{\gamma}} - 1 \right] \right\}^{1/2}$$

An iterative solution is required because the gas properties  $R$ ,  $T$  and  $P$  are functions of a variable  $\gamma$ . If an ideal gas was being considered,  $\gamma$  would be constant and  $v_3$  may be solved for directly.

## 6.5 PARTICLE LIMITING STREAMLINE POINT

Figure 6-6 presents the mesh construction required to solve for the gas and particle properties at any particle limiting streamline point in Figure 6-1.

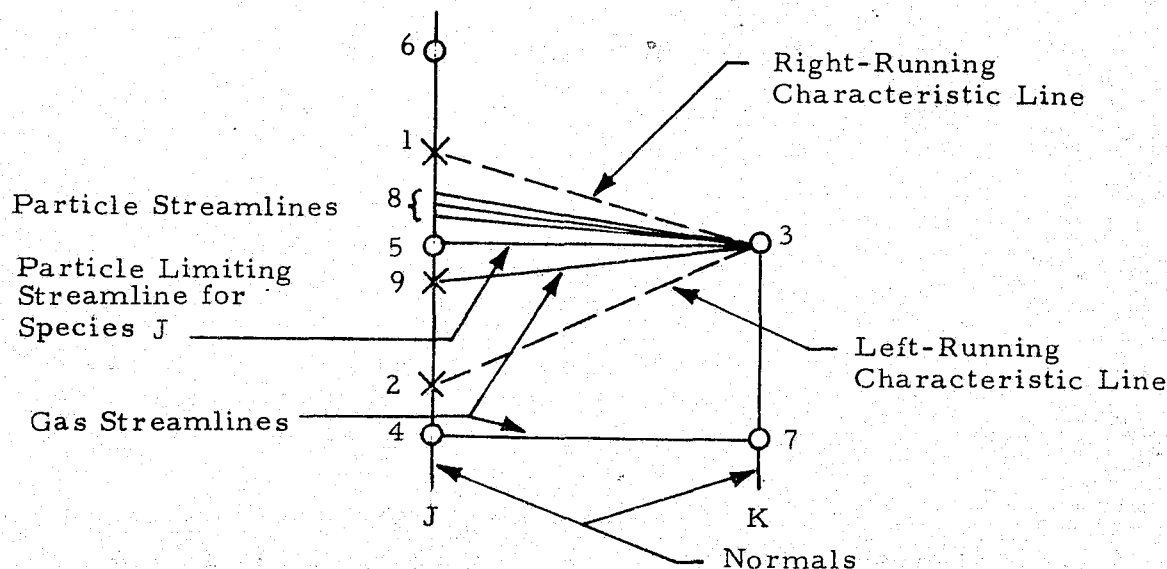


Fig. 6-6 - Mesh Construction for a Particle Limiting Streamline Point

The calculational procedure followed to determine the flow properties at a particle limiting streamline point is somewhat different from that of an interior point solution. The differences in the calculational procedure are brought about by the fact that the  $J$  and  $K$  normals are constructed between a gas streamline and a particle limiting streamline rather than between two gas streamlines.

The physical location  $(x_3, y_3)$  of point  $3K$  is found by intersecting the normal from the point  $7K$  with the projection of the particle limiting streamline of species  $J$  from the point  $5J$ . Once point  $3K$  is located, the gas streamline



intersection with the J data line must be located (point 9 of Figure 6-6) using the gas streamline relationship

$$\frac{y_3 - y_9}{x_3 - x_9} = \frac{v_3}{u_3} \quad (3.41)$$

The properties at this point are found by interpolation between the known points 4J and 5J.

The particle density at point 3K for all particle species except particle species J is determined in the same manner as for an interior point. The particle density of species J is calculated using the relationship

$$\rho_3 = \frac{1}{u_3 (y_3 y_3^\delta - y_2 y_2^\delta) - v_3 (y_3^\delta + y_2^\delta) (x_3 - x_2)} \left\{ \rho_5 \left[ u_5 (y_5 y_5^\delta - y_2 y_2^\delta) - v_5 (y_5^\delta + y_2^\delta) (x_5 - x_2) \right] + \rho_2 \left[ u_2 (y_5 y_5^\delta - y_3 y_3^\delta) - v_2 \left[ (y_5^\delta + y_2^\delta) (x_5 - x_2) - (y_3^\delta + y_2^\delta) (x_3 - x_2) \right] \right] \right\} \quad (3.169)$$

and is based on the fact that the entire mass of particle species J is contained in the streamtube formed by the particle limiting streamline for species J.

Once the particle densities have been found, the calculation procedure is continued as in the case of an interior point.

## 6.6 PARTICLE LIMITING STREAMLINE-BOUNDARY INTERSECTION

Figure 6-7 presents the mesh construction required to solve for the gas and particle properties at the particle limiting streamline-boundary intersection.

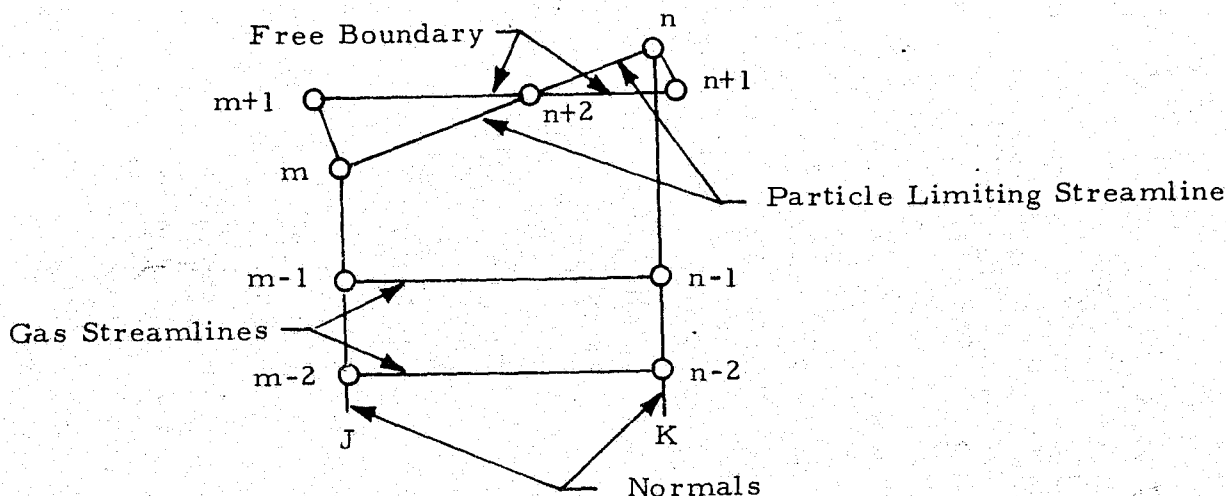


Fig. 6-7 - Mesh Construction for a Particle Limiting Streamline-Boundary Intersection

Assume that the particle limiting streamline point  $(n, K)$  has just been completed and the next point to be calculated is that of a free boundary point  $(n+1, K)$ . Under these circumstances, the free boundary point is calculated in the normal manner, and then upon completion of the free boundary point a comparison is made of the physical location of the two points. If  $y_{n+1} < y_n$ , an intersection of the particle limiting streamline and free boundary is indicated. The physical location of  $(n+2, K)$  is found by intersecting the projection of the streamline from the point  $(m+1, J)$  and the particle limiting streamline between the points  $(m, J)$  and  $(n, K)$ . The gas and particle properties at the point  $(n+2, K)$  are then found by interpolation between the points  $(m, J)$  and  $(n, K)$ .

Once the properties at the point  $(n+2, K)$  are found, the point  $(n+1, K)$  is assumed to no longer exist. The point  $(n+2, K)$  is used as the base point the next time an upper boundary point is calculated.

The description of the entire flow field now proceeds onto the calculation of the lower boundary point of the new K-line.

## 6.7 SHOCK WAVES

Unlike the interior point solution, there are two points assigned to every shock wave point. To determine if the shock wave is either left-running or right-running, one can check the shock points  $(m, J)$  and  $(m-1, J)$  in Figure 6-8. If the point  $(m, J)$  is a shock upstream point and the point  $(m-1, J)$  is a shock downstream point, the shock wave is a left running one as shown in Figure 6-8(a). If the case is reversed, the shock wave is right-running as shown in Figure 6-8(b). Once the shock wave is detected, the new shock point  $(n, K)$  is found by extending the shock wave from point  $(m, J)$  and the normal from point  $(n-1, K)$ . Generally, first, the properties of the shock upstream point are found according to the upstream reference properties and then the properties of the shock downstream point are calculated with the oblique shock relations using Eq. (4.22). The velocity of the shock downstream point is also calculated by using the shock downstream reference properties with one characteristic. The velocity calculated by both methods is compared and the shock angle is adjusted until

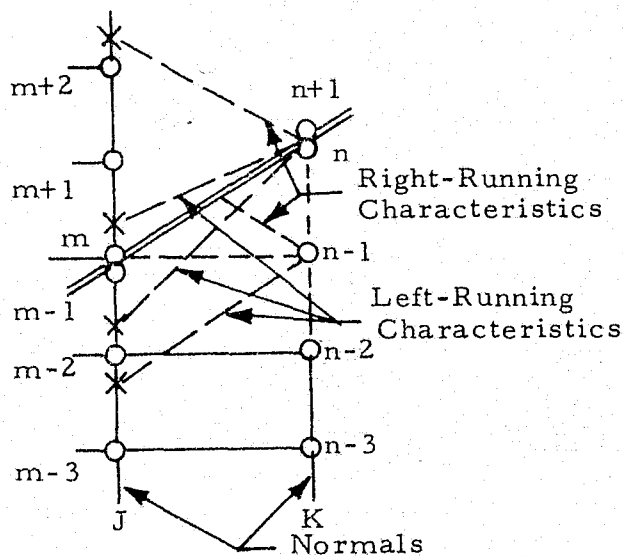


Fig. 6-8a - Mesh Construction for a Left-Running Shock Wave Point

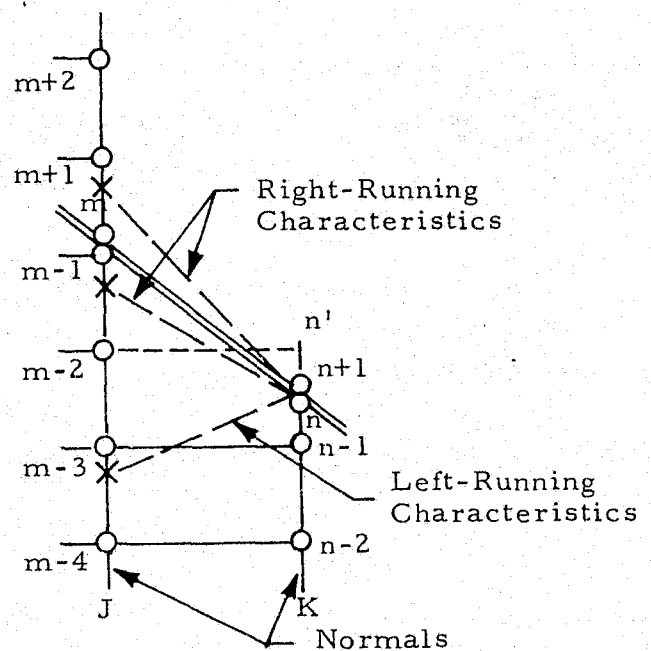


Fig. 6-8b - Mesh Construction for a Right-Running Shock Wave Point

the velocity calculated by both methods satisfies the convergence criteria. (Note that the streamline is generally terminated in front of the shock wave and a new one started from the shock downstream point.)

#### 6.7.1 Left-Running Shock

As shown in Figure 6-8(a), the shock point location is found first and the shock upstream point  $(n+1, K)$  is treated as an interior point. The calculation is carried out as described in the preceding paragraph.

Under normal conditions, the calculation for point  $(n-1, K)$  in Figure 6-8(a) is slightly different from that for the regular interior point. As can be seen, the right-running characteristic from that point intersects the shock wave before intersecting the J-line. The base point properties on the right-running characteristic are interpolated between point  $(m-1, J)$  and  $(n, K)$ . Because of this, points  $(n-1, K)$ ,  $(n, K)$  and  $(n+1, K)$  are handled simultaneously. Under certain conditions, the right-running characteristic from point  $(n-2, K)$  must be added to this group if its right-running characteristic intersects the shock wave before intersecting the J-line.

#### 6.7.2 Right-Running Shock

Assume that a right-running shock is detected when the point  $(n', K)$  as shown in Fig. 6-8(b) is attempted. The calculation for this point is terminated and the intersection of the shock wave with the K-line is found. The shock upstream point  $(n, K)$  is treated as an interior point and the shock points calculation is carried out as usual. Under certain cases, the properties of the shock upstream point are interpolated from the two known points on the K-line if the intersection of the shock wave with the K-line falls between these known points. For instance, if the point  $(n', K)$  in Fig. 6-8(b) is complete, the properties of the shock upstream point  $(n, K)$  are interpolated from points  $(n-1, K)$  and  $(n', K)$  (see Section 6.8.1 Upper Compression Corner).

The calculation of the first few interior points on the K-line behind the right-running shock wave is slightly different from the regular interior point.

As shown in Figure 6-8(c), the left-running characteristic from point  $(n+2, K)$  intersects the shock wave and its base point properties are interpolated between points  $(m, J)$  and  $(n+1, K)$ . The left-running characteristic from point  $(n+3, K)$  would have intersected the shock wave but the base point properties are interpolated between points  $(m, J)$  and  $(n+2, K)$ .

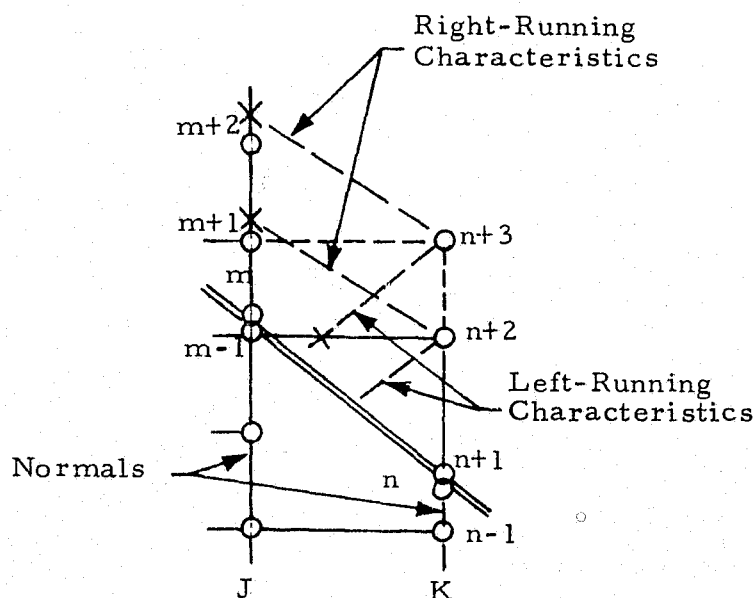


Fig. 6-8c - Mesh Construction for an Interior Point  
Behind a Right-Running Shock Wave

## 6.8 COMPRESSION CORNER POINT

The locations of compression corners in the flow field are defined by the boundary conditions of the problem. Whenever a boundary point is calculated, the location of this point is always compared to the limit of the current boundary equation. If the boundary point is already beyond the limit, the properties of the limiting point are interpolated between the boundary point on the J-line and the newly calculated boundary point whose location has exceeded the limit of the current boundary. The calculation thereafter will depend on the type of discontinuity and the type of following boundary.

### 6.8.1 Upper Compression Corner

Figure 6-9a shows a typical compression corner at the upper boundary. The interior point  $(n', K)$  and the upper boundary point  $(n'+1, K)$  are first calcu-

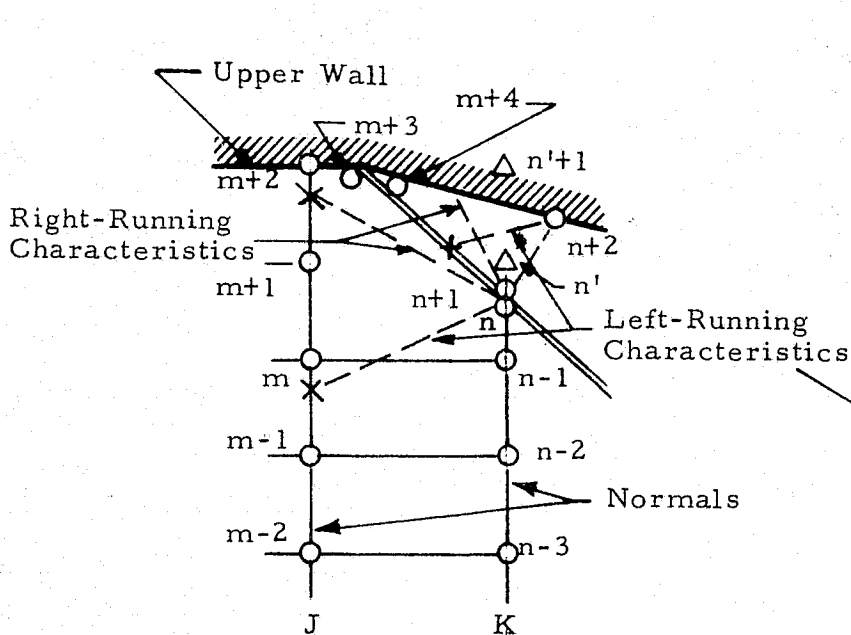


Fig. 6-9a - Mesh Construction for an Upper Compression Corner Point

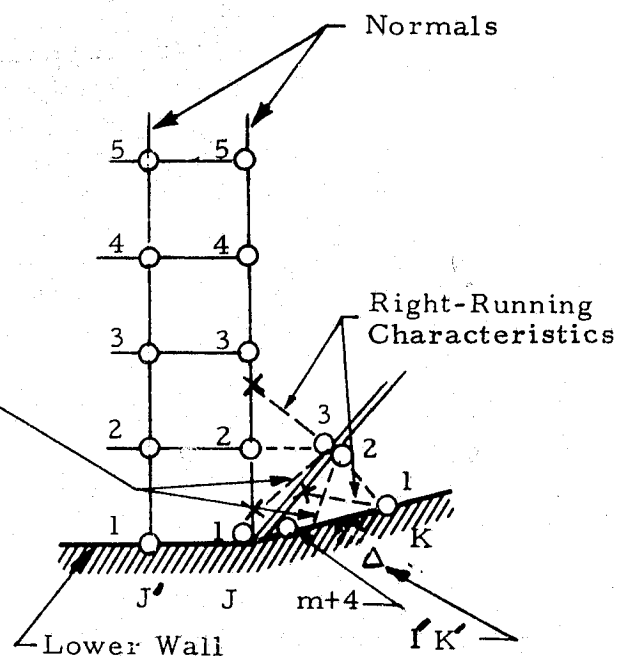


Fig. 6-9b - Mesh Construction for a Lower Compression Corner Point

lated without any knowledge of the existence of the compression discontinuity. Then the properties at the limiting point  $(m+3, J)$  are found by interpolation between the points  $(m+2, J)$  and  $(n'+1, K)$ . Since the boundaries are already given, the amount of turn the flow must make can be found directly from these boundaries. With the information of the known turning angle and the flow properties at the corner, point  $(m+3, J)$ , a right-running shock is defined and the properties of the shock downstream point  $(m+4, J)$  can be calculated with the oblique shock relations using Eq. (4.22). After the shock wave is established, it is extended from the discontinuity corner to intersect the just calculated K-line. The flow properties at the intersection point  $(n, K)$  in Fig. 6-9a, are interpolated between points  $(n-1, K)$  and  $(n', K)$  which are already calculated. The properties

thus obtained are for the shock upstream point  $(n, K)$ . The calculation for the shock downstream point and the boundary point is as follows. The properties of the shock downstream point  $(n+1, K)$  are first calculated using the oblique shock relations. A normal is then drawn from point  $(n+1, K)$  to intersect the new boundary at point  $(n+2, K)$ . The flow properties at point  $(n+2, K)$  are at first assumed to be the same as those of point  $(m+4, J)$ . Note that, by doing so, the entropy of point  $(n+2, K)$  is automatically set equal to that at point  $(m+4, J)$ . The velocity at point  $(n+2, K)$  is then calculated by using the left-running characteristic with its base point properties interpolated between points  $(m+4, J)$  and  $(n+1, K)$ . The other properties at point  $(n+2, K)$  are also adjusted accordingly. The velocity at point  $(n+1, K)$  is then calculated by using the right-running characteristic with its base point properties interpolated between points  $(m+4, J)$  and  $(n+2, K)$ . The flow velocity thus calculated is compared with that calculated from the oblique shock relations based on the shock upstream properties. The shock angle at the K-line is adjusted and the calculation repeated until the velocity calculated with both methods satisfies the convergence criteria. At the beginning of every new calculation cycle, point  $(n, K)$  is relocated with the new average shock angle between the points  $(m+3, J)$  and  $(n, K)$ , and the new average normal between the points  $(n-1, K)$  and  $(n, K)$ . The average normal is also used to locate point  $(n+2, K)$ .

### 6.8.2 Lower Compression Corner

Figure 6-9b shows a typical compression corner at the lower boundary. Assume that the J'-line has just been calculated. The lower boundary point  $(l', K')$  is first calculated without any knowledge of the existence of the compression discontinuity. The flow properties at the compression corner are then calculated in the same manner as for the upper compression corner and a left-running shock wave is established. A new J-line in the shock upstream region is created emanating from the lower compression corner.

To start a new K-line in the shock downstream region corresponding to the new J-line in the shock upstream region, the following method is adopted. The intersection of the shock wave from point  $(l, J)$  and the streamline (if the shock

angle is very small, the right-running characteristic is used) from point (2, J) is found first. This point, the shock upstream point (3, K), is treated as an interior point and the oblique shock relations are used to calculate the properties of the shock downstream point (2, K). An average normal is drawn from this point to intersect the new lower boundary, which is designated as (1, K) in Figure 6-9(b). The properties at point (1, K) are initially set equal to those at point (m+4, J); its velocity is recalculated by using the right-running characteristic with its base point properties interpolated between points (2, K) and (m+4, J). The other properties at point (1, K) are adjusted accordingly. The velocity at point (2, K) is then calculated with the aid of the left-running characteristic with its base point properties interpolated between points (m+4, J) and (1, K). The rest of the calculation is carried out in the same way as in the case for the upper compression corner.

## 6.9 EXPANSION CORNER - PRANDTL-MEYER FAN

Expansion corners are found when the slope of the solid boundary has a discontinuity in such a way that the flow must negotiate a sharp turn away from the oncoming flow or when the flow issuing into an open space is under-expanded in a nozzle or a flow channel. The expansion takes place either in a channel or at the channel exit. The amount of turn the flow must make and the final flow velocity can be found with the aid of the Prandtl-Meyer expansion relation, Eq. 5-1, and the boundary conditions.

After the total turning angle is found, it is subdivided into a number of small turning angles. The flow properties at the expansion corner corresponding to each of the small turns are calculated and the angle's corresponding characteristic lines emanating from the corner are constructed. The final line of the expansion is a streamline rather than a characteristic line and, normally, the Mach angle corresponding to this streamline is substantially greater than the small turning angle. An extra point is inserted to fill the gap after the expansion corner computation has been completed. Note that this final expansion line, a streamline, can be a solid boundary or a free boundary.



In the example shown in Figure 6-10, the turning angle, is subdivided into four small turns. The corresponding characteristic lines emanating from the corner are designated as A, B, C and D in Figure 6-10. The final expansion line E is a streamline corresponding to the last corner point (d, J). The properties at the corner corresponding to every small turn are stored in (a, J), (b, J), (c, J) and (d, J) arrays. The properties at the corner before the expansion takes place can be found as usual and are stored in (m, J) array; its characteristic line is designated as M in Figure 6-10.

A weak shock is initialized at point (n+4, K) for the upper corner expansion point or (n-4, K) for the lower corner expansion, which is done simply by defining a double point at those points with identical flow properties. The initial shock angle is assumed equal to the Mach angle.

#### 6.9.1 Upper Corner Expansion

As shown in Figure 6-10(a), the point (n-1, K) is considered to be complete. A left-running characteristic is drawn from the point (n-1, K) to intersect the right-running characteristic from point (m, J) at M, which is an interior point. The properties can be calculated in the usual manner. A normal is drawn from point (n-1, K) to intersect the right-running characteristic from point (m, J) at n. The properties at this point are then interpolated between points (m, J) and M. Similarly, the properties at points (n+1, K) through (n+4, K) can be calculated. Point (n+5, K) is calculated by the same technique provided that the point E is treated as an upper boundary point. If the distance between points (n+4, K) and (n+5, K) is greater than other meshes in the expansion fan, one extra point (n+6, K) is inserted and its properties are found by interpolation between points (n+4, K) and (n+5, K).

#### 6.9.2 Lower Corner Expansion

This case is shown in Figure 6-10(b). When an expansion at the lower boundary is detected, the normal line starting from the corner point (m, J) is completed first. The flow properties at the corner and the corresponding

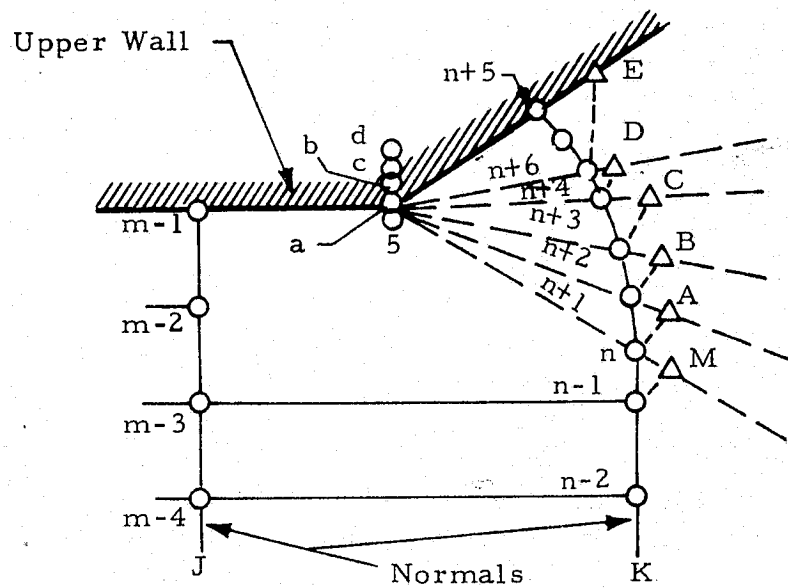


Fig. 6-10a - Mesh Construction for an Upper Expansion Corner Point

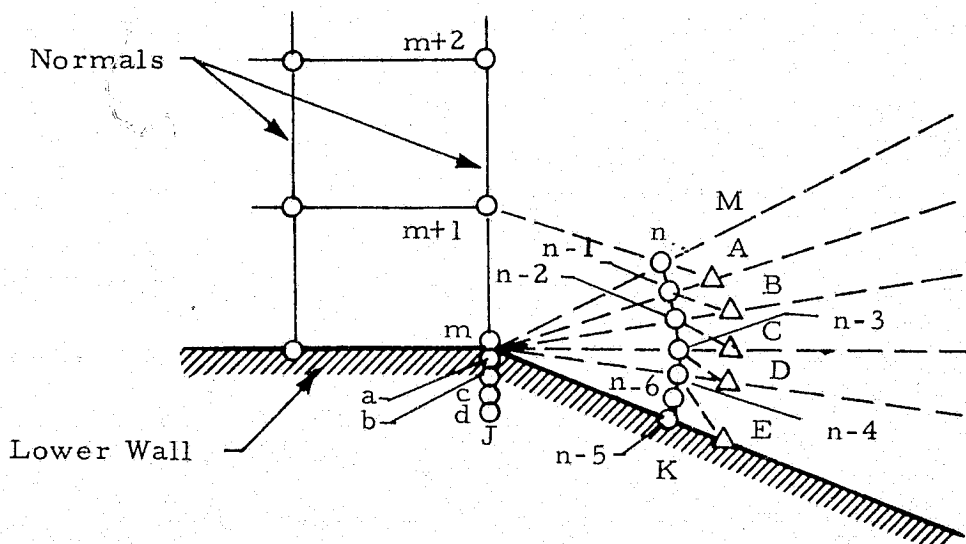


Fig. 6-10b - Mesh Construction for a Lower Expansion Corner Point

characteristic line for each small turn can be found by the same method described in the preceding paragraph. Point  $(n, K)$  is the intersection of the right-running characteristic from point  $(m+1, J)$  and the left-running characteristic from point  $(m, J)$ ; its properties can be calculated without problem. The flow properties at point A can also be calculated. A normal from point  $(n, K)$  and the left-running characteristic from point  $(a, J)$  intersects at  $(n-1, K)$ ; flow properties at this point are then interpolated between points  $(a, J)$  and A. Points  $(n-2, K)$  through  $(n-4, K)$  can be calculated by the same method. Point  $(n-5, K)$  is a lower boundary point which can be calculated with the technique used to find the point  $(n+5, K)$  in Figure 6-10(a). An extra point may also be inserted between points  $(n-4, K)$  and  $(n-5, K)$  if needed.

#### 6.10 SLIP LINE POINT

Sliplines are normally found behind shock wave intersections and in multiple streamtube flow where each streamtube retains its own identity. For every slipline in the flow field, two points are assigned and initialized at the input line. The conditions which the slipline points have to satisfy are that the flow angle and the pressure on both sides of the slipline must be the same.

The point  $(n, K)$  in Figure 6-11(a) is found by extending the slipline from point  $(m, J)$  and the normal from point  $(n-1, K)$  as usual. First, the flow angle at the new slipline points is assumed to be identical to that of point  $(m, J)$ , and the velocity at points  $(n, K)$  and  $(n+1, K)$  are evaluated by using II- and I-characteristic respectively. Then, the pressure at both sides of the slipline is calculated and compared. If a pressure difference exists, the flow angle is adjusted back and forth until the pressure is balanced on both sides of the slipline.

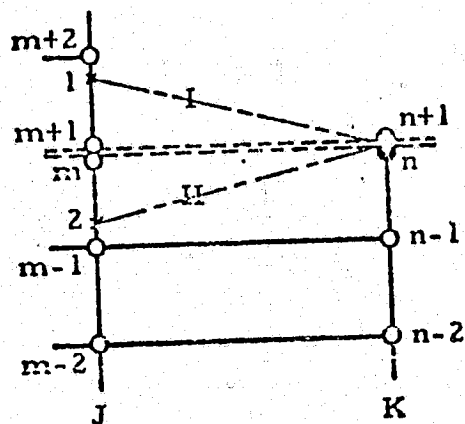


Fig. 6-11a - Mesh Construction for a Slip Line Point

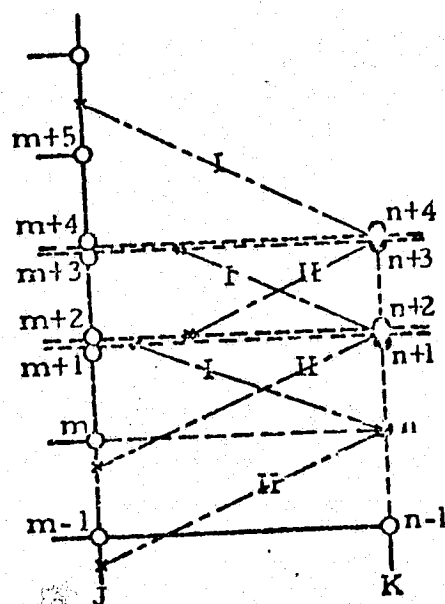


Fig. 6-11b - Mesh Construction for a Slip Line Point Where a Characteristic Intersects a Slip Line

During the calculation, crossings of the characteristic line with the slip-line are not allowed. For example in Figure 6-11(b), the I-characteristic of point (n, K) intersects the slipline first before intersecting the J-line. The reference properties for the calculation of point (n, K) on I-characteristic are interpolated between points (m+1, J) and (n+1, K) if point (n+1, K) is known; when the point (n+1, K) is still unknown, the properties at point (m+1, J) are used. The slipline points (n+1, K), (n+2, K) and (n+3, K), (n+4, K) can be handled as described in the preceding paragraph. For the case shown in Figure 6-11(b), the points (n, K) (n+1, K), (n+2, K), (n+3, K) and (n+4, K) are iterated as a group, since the calculation of any point in this group is interrelated.

## Section 7

### NUMERICAL INTEGRATION OF THE CONSERVATION EQUATIONS

Examination of the mass flow rate, momentum flux and energy flux through a nozzle and exhaust plume will yield considerable information about the nozzle performance and the status of the flowfield numerical solution. The momentum flux is used to determine the thrust produced by the nozzle while the thrust ratioed to the mass flow rate is a measure of the nozzle performance. Deviations of the fluxes in the conservative parameters from surface to surface (Fig. 7-1) are indicators of the validity of a numerical solution such as the one used in the gas-particle code.

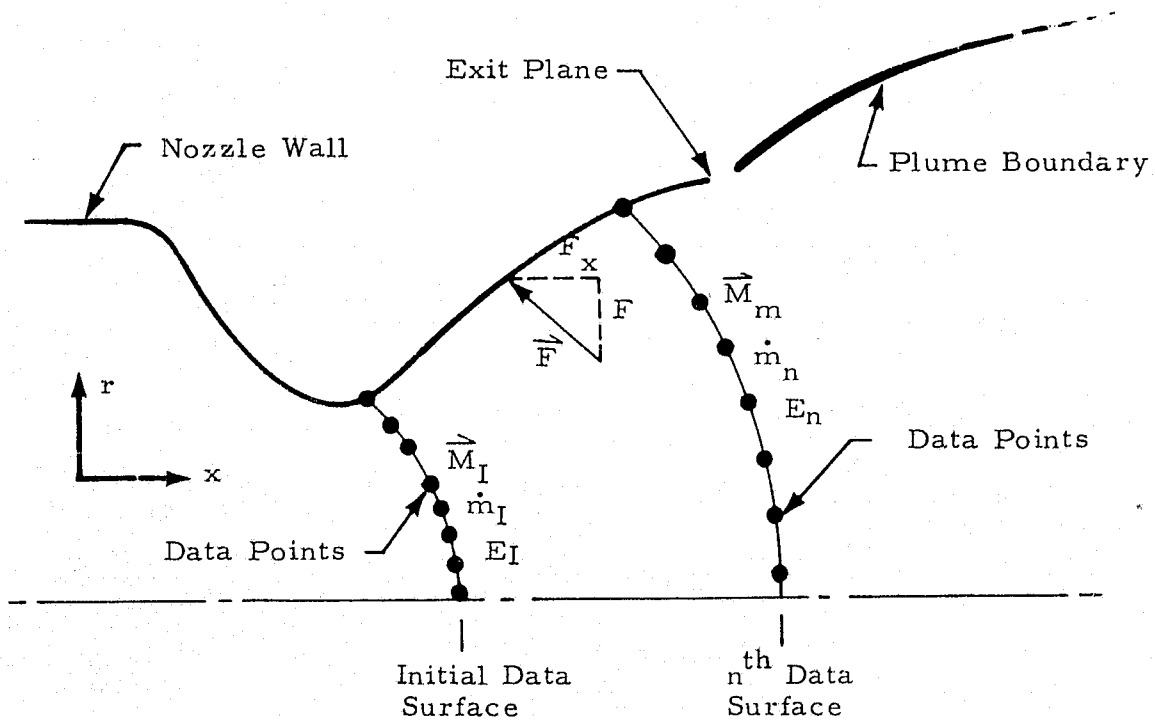


Fig. 7-1 - Schematic of Conservation Parameters in a Nozzle Plume

## 7.1 CONTINUITY

The mass flow rate through a surface is calculated by numerically integrating the incremental mass flow through adjacent streamtubes. A streamtube is defined as the flow area bounded by adjacent data points (Fig. 7-2a) and the incremental flow rate is calculated as follows.

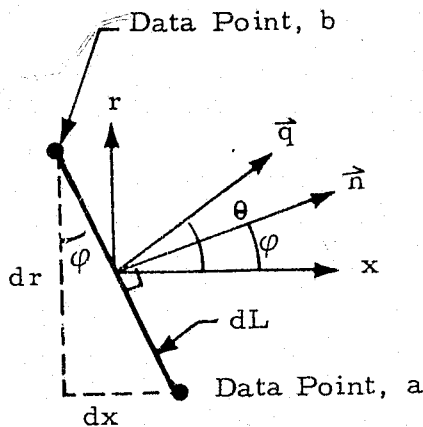


Fig. 7-2a - Schematic of Surface Geometry

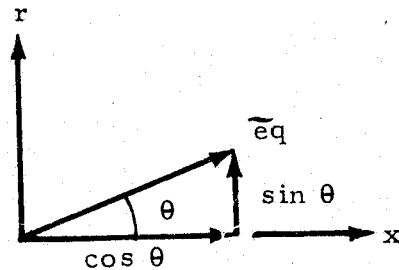


Fig. 7-2b - Schematic of Velocity Unit Vector

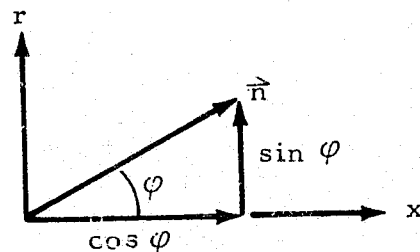


Fig. 7-2c - Schematic of Surface Normal Unit Vector

Fig. 7-2 - Schematic of Elemental Geometry on a Data Surface

Let points a and b be any two points on a data surface within the flow field. The incremental flow area for axisymmetric or two-dimensional flow is given by

$$dA = (2\pi r)^\delta dL. \quad (7.1)$$

In terms of the coordinate directions, the incremental area is

$$dA = (2\pi r)^\delta \{ (dr)^2 + (dx)^2 \}^{1/2} \quad (7.2)$$

The flow area in the direction normal to the surface,  $dL$ , is now

$$\vec{dA} = (2\pi r)^\delta \{ (dr)^2 + (dx)^2 \}^{1/2} \hat{e}_n \quad (7.3)$$

where the unit normal vector,  $\hat{e}_n$ , is defined by the geometry of Fig. 7-2c as

$$\hat{e}_n = \cos\varphi \hat{i} + \sin\varphi \hat{j}; \quad (7.4)$$

and the angular location of the unit normal vector relative to the x-axis is

$$\varphi = -\tan^{-1} \left( \frac{dx}{dr} \right) \quad (7.5)$$

The quantity of mass flowing through the incremental area is found from the relation

$$d\dot{m} = \rho |\vec{q}| dA \hat{e}_q \cdot \hat{e}_n; \quad (7.6)$$

where the unit velocity vector,  $\hat{e}_q$  is defined by the geometry of Fig. 7-2b. The relation for  $\dot{m}$  is now

$$\dot{m} = \rho |\vec{q}| dA \{ \cos\theta \hat{i} + \sin\theta \hat{j} \} \cdot \{ \cos\varphi \hat{i} + \sin\varphi \hat{j} \}.$$

Performing the indicated dot product the relation reduces to

$$\dot{m} = \rho |\vec{q}| dA \{ \cos\theta \cos\varphi + \sin\theta \sin\varphi \}.$$

The trigometric identity for sine-cosine function is

$$\cos\theta \cos\varphi + \sin\theta \sin\varphi = \cos(\theta - \varphi).$$

Substituting, the incremental mass flow is

$$dm = \rho |\vec{q}| dA \cos(\theta - \varphi) . \quad (7.7)$$

The total mass flow through the surface is simply

$$\dot{m} = \sum dm \quad (7.8)$$

for all the incremental flow areas.

So far the discussion has been directed toward a general expression for the mass flow rate. For gas-particle flows the surface geometry is the same. The only change is in the definition of the flow variables. The expression for the respective phases are as follows:

#### Gas Phase

$$\dot{m} = \rho |\vec{q}| dA \cos(\theta - \varphi) ; \quad (7.9a)$$

#### Particle ( $j^{\text{th}}$ particle)

$$\dot{m}^j = \rho^j |\vec{q}^j| dA \cos(\theta^j - \varphi) ; \quad (7.9b)$$

#### Mixture

$$\dot{m}_m = \dot{m} + \sum_{j=1}^{NP} \dot{m}^j \quad (7.9c)$$

## 7.2 MOMENTUM

As with the mass flow relations, a general expression for the momentum flux will be developed. This expression will then be extended to include the treatment of gas-particle flows.



The general expression for the momentum flux is

$$\vec{M}_M = \left\{ P \vec{\delta} + \rho \vec{q} \vec{q} \right\} \cdot d\vec{A} \quad (7.10)$$

where  $\vec{\delta}$  is the Kroneker delta and  $d\vec{A}$  is given by Eq. (7.3). Substituting for  $d\vec{A}$  and performing the indicated algebraic operations, Eq. (7.10) becomes

$$\vec{M}_M = \left\{ P \vec{n} + \rho \vec{q} \vec{q} \cdot \vec{n} \right\} dA \quad (7.11)$$

where  $dA$  is now defined by Eq. (7.2). Equation (7.11) yields an expression for each coordinate direction in the equation in terms of the scalar components given by the expressions

$$\begin{aligned} \vec{M}_M &= \left\{ P(\cos\phi \hat{i} + \sin\phi \hat{j}) + \rho(u\hat{i} + v\hat{j})(u\hat{i} + v\hat{j}) \cdot (\cos\phi \hat{i} + \sin\phi \hat{j}) \right\} dA; \\ \vec{M}_M &= \left\{ P(\cos\phi \hat{i} + \sin\phi \hat{j}) + \rho(u^2 \hat{i}\hat{i} + uv\hat{i}\hat{j} + uv\hat{j}\hat{i} + v^2 \hat{j}\hat{j}) \cdot (\cos\phi \hat{i} + \sin\phi \hat{j}) \right\} dA; \\ \vec{M}_M &= \left\{ P(\cos\phi \hat{i} + \sin\phi \hat{j}) + \rho(u^2 \cos\phi \hat{i} + uv \cos\phi \hat{j} + uv \sin\phi \hat{i} + v^2 \sin\phi \hat{j}) \right\} dA. \end{aligned}$$

The scalar equations in the coordinate directions are then

$$M_{M_x} = \left\{ P \cos\phi + \rho(u^2 \cos\phi + uv \sin\phi) \right\} dA,$$

and

$$M_{M_y} = \left\{ P \sin\phi + \rho(uv \cos\phi + v^2 \sin\phi) \right\} dA;$$

Substituting for the velocity components

$$M_{M_x} = \left\{ P \cos\phi + \rho q^2 (\cos\theta \cos\theta \cos\phi + \cos\theta \sin\theta \sin\phi) \right\} dA$$

and

$$M_{M_y} = \left\{ P \sin\phi + \rho q^2 (\cos\theta \sin\theta \cos\phi + \sin\theta \sin\theta \sin\phi) \right\} dA$$

Factoring common terms and substituting the trigometric identity as before the final result for the incremental momentum flux is

$$M_{M_x} = \left\{ P \cos \varphi + \rho q^2 \cos(\theta - \varphi) \cos \theta \right\} dA, \quad (7.12a)$$

and

$$M_{M_r} = \left\{ P \sin \varphi + \rho q^2 \cos(\theta - \varphi) \sin \theta \right\} dA \quad (7.12b)$$

The only difference between the expression for the gas and particle contributions to the momentum flux is that the particles do not contribute to the pressure of the mixture. The respective expressions for the momentum flux through the surface are:

#### Gas

$$M_{M_x} = \left\{ P \cos \varphi + \rho q^2 \cos(\theta - \varphi) \cos \theta \right\} A \quad (7.13a)$$

$$M_{M_r} = \left\{ P \sin \varphi + \rho q^2 \cos(\theta - \varphi) \sin \theta \right\} A \quad (7.13b)$$

#### Particles ( $j^{\text{th}}$ Particle)

$$M_{M_x}^j = \rho^j (q^2)^j \cos(\theta - \varphi) \cos(\theta^j) A \quad (7.13c)$$

$$M_{M_r}^j = \rho^j (q^2)^j \cos(\theta - \varphi) \sin(\theta^j) A \quad (7.13d)$$

#### Mixture

$$M_{M_{m_x}} = M_{M_x} + \sum_{j=1}^{NP} M_{M_x}^j \quad (7.14a)$$

$$M_{M_{m_r}} = M_{M_r} + \sum_{j=1}^{NP} M_{M_r}^j \quad (7.14b)$$

### 7.3 ENERGY

The general relation is

$$E = \left(h + \frac{1}{2} q^2\right) \dot{m} \quad (7.15)$$

Applied to gas and particulate phases the respective relations for a given surface are:

#### Gas

$$E = \left(h + \frac{1}{2} q^2\right) \dot{m} \quad (7.16a)$$

#### Particles ( $j^{\text{th}}$ Particle)

$$E^j = \left(h^j + \frac{1}{2} (q^2)^j\right) \dot{m}^j \quad (7.16b)$$

#### Mixture

$$E_m = E + \sum_{j=1}^{NP} E^j \quad (7.16c)$$

### 7.4 NOZZLE PERFORMANCE

Performance parameters considered are the thrust produced and the associated specific impulse,  $I_{sp}$ . The force vector is calculated as the mixture momentum on the initial data surface (Fig. 7-1) plus the pressure force on the nozzle wall. Mathematically this is expressed as

$$\vec{F}_T = \vec{M}_{m_I} + \int P_w \vec{dA}_w \quad (7.17)$$

The integral term in Eq. (7.17) is obtained by numerically integrating the pressure force along the nozzle wall. Consider the elemental surface,  $dL$ , bounded by the nozzle wall points  $a$  and  $b$  on any two successive data

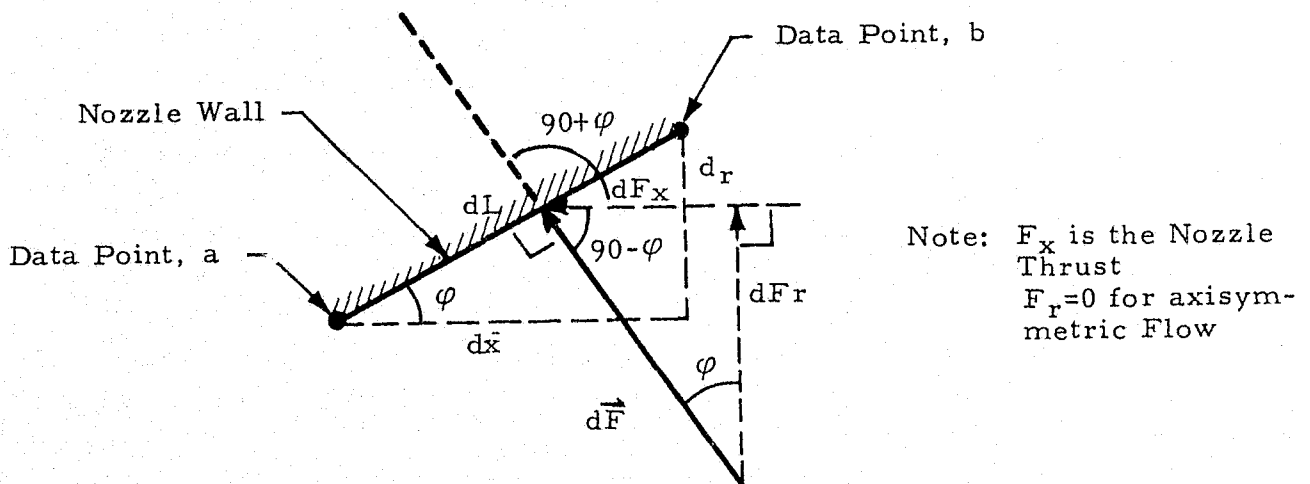


Fig. 7-3 - Schematic of Pressure Force Acting on the Nozzle Wall

surfaces. The slope of the surface,  $dL$ , is given by

$$\varphi_L = \tan^{-1} \left( \frac{dr}{dx} \right) . \quad (7.18)$$

However, the force due to the pressure loading acts normal to the surface which is given by the angular relation  $90 + \varphi$ . The local surface area over which the pressure is acting is

$$dA = (2\pi r)^\delta \left\{ (dr)^2 + (dx)^2 \right\}^{1/2} ; \quad (7.19)$$

and the unit normal to the surface is

$$\vec{n}_L = -\sin\varphi \hat{i} + \cos\varphi \hat{j} , \quad (7.20)$$

so that

$$d\vec{A} = dA \vec{n}$$

$$d\vec{A} = (2\pi r)^\delta \left\{ (dr)^2 + (dx)^2 \right\}^{1/2} \left\{ -\sin\phi \hat{i} + \cos(\phi) \hat{j} \right\} \quad (7.21)$$

Equation (7.21) can be generalized to reflect the loading increment to either an upper or lower wall in the following manner. Define a parameter  $\lambda$  which has a value of  $\pm 1$  for an upper or lower wall, respectively. The orientation of the surface unit normal with respect to the x-axis is

$$\phi_1 = \phi_L + \lambda 90 \quad (7.22)$$

From the sine and cosine trigometric identities Eq. (7.22) reduces to

$$\cos\phi_1 = -\sin\phi$$

and

$$\sin\phi_1 = \cos\phi$$

so that the unit normal vector is now given by

$$\vec{n} = \cos\phi_1 \hat{i} + \sin\phi_1 \hat{j};$$

and the surface area term is now

$$d\vec{A}_w = (2\pi r)^\delta \left\{ (dr)^2 + (dx)^2 \right\}^{1/2} \left\{ \cos(\phi_1) \hat{i} + \sin(\phi_1) \hat{j} \right\}$$

The resultant force relation for the nozzle is then

$$\vec{F}_T = \vec{M}_{m_I} + \sum_{i=1}^{NS} \left[ P (2\pi r)^\delta \left\{ (dr)^2 + (dx)^2 \right\}^{1/2} \left\{ \cos(\phi_1) \hat{i} + \sin\phi_1 \hat{j} \right\} \right] \quad (7.23)$$

and the nozzle thrust is just  $F_x$ . The specific impulse,  $I_{sp}$ , is then

$$I_{sp} = F_x / \dot{m}_{m_I} \quad (7.24)$$

## 7.5 PERCENT CHANGES IN THE CONSERVATION QUANTITIES

One check on the validity of the numerical solution is to monitor the change in the conservation quantities mass flow, momentum and energy. If these quantities do not change relative to the initial surface values, then the mass, momentum and energy have been conserved; this is a good indication that the numerical solution is proceeding satisfactorily. However, since a numerical solution is employed some changes in the conservation quantities is to be expected. The objective then is to keep the error as small as possible. When large errors occur then the input data, wall equations, mesh construction, etc., should be checked.

The changes in the conservation quantities are found by comparing the integrated value at a given data surface relative to the initial data surface. These are as follows:

### Mass Flow Rate (Mixture):

$$\% \Delta \dot{m} = \frac{(\dot{m}_N - \dot{m}_I) \cdot 100}{\dot{m}_I} \quad (7.25)$$

### Momentum (Mixture)

$$\% \Delta |M| = \frac{(|\vec{M}_N| - |\vec{M}_I|) \cdot 100}{|\vec{M}_I|} \quad (7.26)$$

### Energy (Mixture)

$$\% \Delta E = \frac{(E_N - E_I) \cdot 100}{E_I} \quad (7.27)$$

The individual changes for the respective phases are computed in a similar fashion as specified by Eqs. (7.25) through (7.27). One important item to remember is that when gas particle flows are being considered the

solution is coupled in the sense that momentum and energy is exchanged between the phases. This means that relative large changes in the momentum and energy quantities for each phase can occur. What should be of concern for gas-particle flows is the change in the mixture conservation quantities (momentum and energy). The respective mass flows should be conserved since the assumption has been made that the phases do not exchange mass via chemical reactions, condensation, etc.

Finally, the specific impulse for a nozzle can be calculated in one of several ways. One way has already been outlined where the thrust is computed as the sum of the initial surface momentum and the pressure force acting on the nozzle wall. The specific impulse is then calculated by dividing the thrust by the mass flow rate through the initial surface. Using the initial surface mass flow rate is a matter of choice and can be argued from either a pro or con view. However, use of the initial surface mass flow rate eliminates the use of one variable which would be numerically calculated. Computation of the specific impulse by the second procedure is as follows.

The thrust generated by the nozzle is first obtained by calculating the momentum in the nozzle axial direction ( $M_{m_x}$ , Eq. (7.14a)). The specific impulse is then calculated by dividing by the mixture mass flow rate. This gives a means to compare the  $I_{sp}$  calculation previously discussed. The change in  $I_{sp}$  is

$$\% \Delta I_{sp} = \frac{(M_{m_x} / \dot{m}_m - I_{sp}) * 100}{I_{sp}} \quad (7.28)$$

where the term  $I_{sp}$  is obtained from Eq. (7.24).

## Section 8

### CONCLUDING REMARKS

The preceding sections have presented a detailed development and description of a numerical solution for the treatment of supersonic gas-particle flows. A computer code has been developed and is described in Volume II of this report. Several options have been included that necessitated a somewhat generalized set of input data. These options include:

- Gas-Particle Flows
- Chemical
  - Equilibrium
  - Kinetics
  - Chemically Frozen
  - Constant Thermodynamics
- Single Phase Solution
  - Non-Isoenergetic Flow
- Non-Continuum Flow
- Performance Calculations

A primary consequence of this work is the extension of gas-particle solutions to treat chemical kinetics for nozzle-exhaust plume flow fields. Since the code has the option of treating single phase flow, chemical kinetics can also be included in liquid propellant motor analyses as well as solid propellant calculations. The method by which the kinetic equations are modeled also permits thermal nonequilibrium to be treated.



In its present form, the code has the capability of producing data for the following applications:

- Gas/Gas-Particle Impingement (Heat Transfer-Loads)
- Rocket Nozzle Performance (Thrust,  $I_{sp}$ )
- IR Signatures (Radiating Species)
- RF Attenuation (Electron Densities)
- Plume Radiation (Radiative Heat Transfer Gas/Particles)
- Vehicle Base Pressure
- Base Heating (Convection-Recirculation)

Future development may be pursued toward developing the capability of handling imbedded subsonic regions (Mach disk) and mixing.

Section 9  
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Appendix A

DISCUSSION OF PARTICLE DRAG AND  
HEAT TRANSFER COEFFICIENTS

# Appendix A SYMBOLS AND NOTATION

<u>Symbol</u>	<u>Description</u>
$C_D$	drag coefficient, dimensionless
$f$	incompressible drag coefficient defined by Eq. (A-2)
$Kn$	Knudsen number, dimensionless
$M$	Mach number, dimensionless
$Nu$	Nusselt number, dimensionless
$Pr$	Prandtl number, dimensionless
$q$	velocity, ft/sec
$r$	particle radius, ft
$R$	gas constant, $\text{ft}^2/\text{sec}^2/^\circ\text{R}$
$Re$	Reynolds number, dimensionless
$T$	temperature, $^\circ\text{R}$
<u>Greek</u>	
$\gamma$	specific heat ratio, dimensionless
$\mu$	viscosity, lbf-sec/ft <sup>2</sup>
$\rho$	density, slug/ft <sup>3</sup>
<u>Subscripts</u>	
$C$	indicates the quantity pertains to the continuum flow regime
$FM$	indicates the quantity pertains to the free molecular flow regime
$I$	indicates the quantity pertains to incompressible flow
<u>Superscript</u>	
$j$	indicates the quantity pertains to a particle species

## APPENDIX A

Particle drag and heat transfer quantities are calculated in terms of non-dimensional coefficients. These coefficients are functions of the local Reynolds number or the Knudsen number depending on the parameter being calculated. The following discussion summarizes the calculations employed in the gas-particle flow analyses. Details of the theory are given in the cited references.

### A.1 PARTICLE DRAG COEFFICIENTS

Particles in solid rocket motor nozzle and exhaust plume flow-fields encounter a wide range of flow regimes. Consequently any expression chosen for the drag coefficient must have the ability to take this into account. In a rocket nozzle and exhaust plume flow field the particles are continually being accelerated by the turbulent flow which has a temperature different than the particles. This situation is further complicated by rarefaction effects because the mean free path in the gas is comparable to dimensions of the particle boundary layer. Obviously the "standard drag curve" of a sphere as a function of Reynolds number (Ref. A-1) does not fit this description. This drag coefficient can only be applied to

- Single solid sphere
- Constant velocity
- Still, isothermal, incompressible flow of infinite extent

In early two-phase studies the sphere drag coefficient was taken as the Stokes drag coefficient;

$$C_D = \frac{24}{Re} \quad (A-1)$$

This expression, however, is valid only for  $Re \leq 1$  if rarefaction effects are not taken into account.

The Stokes drag coefficient is now used as a reference drag coefficient. An incompressible drag parameter defined by the relation

$$f^j = C_D^j / C_{D\text{Stokes}}^j; j=1, NP \quad (A-2)$$

was tabulated as a function of Reynolds number by Kliegel (Ref. A-2). This was corrected for rarefaction effects (Ref. A-3) by the relation

$$C_D = C_{DI} \left\{ \frac{(1 + 7.5 Kn)(1 + 2 Kn) + 1.91 Kn^2}{(1 + 7.5 Kn)(1 + 3 Kn) + (2.29 + 5.16 Kn) Kn^2} \right\} \quad (A-3)$$

The Knudsen number,  $Kn$ , (a measure of flow rarefaction) is related to the Mach and Reynolds numbers by

$$Kn^j = 1.26 \sqrt{\gamma} M^j / Re^j, \quad (A-4)$$

$$M^j = |\Delta q^j| / \sqrt{\gamma RT} \quad (A-5)$$

A more recent formulation for the drag coefficient is taken from the work of Crowe (Ref. A-4). The drag coefficient is defined as

$$C_D = (C_{DFM} - C_{DI}) \bar{C}_D + C_{DI} ;$$

where

$$\bar{C}_D = \left\{ \frac{1.1}{1 + 1.1 (Kn)^{-0.3}} \exp(-Kn^{1/2}) \right\} \left\{ 1 - \exp \left[ -Kn^{.7} e^{\frac{Kn(C_{D_{inc}} - .48)}{6.6}} Re \right] \right\}$$

## A.2 PARTICLE HEAT TRANSFER COEFFICIENTS

At present there appears to be no specific experimental heat transfer data for spheres of the particle sizes in the flow regimes encountered in a rocket nozzle-exhaust plume flow field. A review of semi-empirical expressions is given by Miller and Barrington (Ref. A-5). Until further work is done that is directly applicable to particle flows the continuum expression of Drake (Ref. A-6)

$$Nu_{cont}^j = 2 + 0.459 (Re^j)^{0.55} Pr^{0.333} ;$$

where

$$Re^j = 2\rho |\Delta \bar{q}| r^j / \mu$$

modified for rarefaction effects by Kawanau (Ref. A-6)

$$Nu = \frac{Nu_{cont}}{1.0 + 3.42 M / Re^j Pr Nu_{cont}}$$

appears to be the most applicable to two-phase flows.



Section A-1  
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Appendix B  
NON-ISOENERGETIC GAS PHASE  
FLOW TREATMENT

## Appendix B SYMBOLS AND NOTATION

<u>Symbol</u>	<u>Description</u>
$h$	local enthalpy, $\text{ft}^2/\text{sec}^2$
$H$	total enthalpy, $\text{ft}^2/\text{sec}^2$
$P$	pressure, $\text{lbf}/\text{ft}^2$
$q$	velocity, $\text{ft}/\text{sec}$
$s$	entropy, $\text{ft}^2/\text{sec}^2/^\circ\text{R}$
$T$	temperature, $^\circ\text{R}$
<u>Greek</u>	
$\eta$	weight flow ratio of oxidizer to fuel, dimensionless
$\rho$	density, $\text{slug}/\text{ft}^3$
$\Delta$	denotes a change in a variable over a step length
<u>Subscripts</u>	
$o$	indicates the quantity pertains to the oxidizer
$f$	indicates the quantity pertains to the fuel
<u>Superscript</u>	
$\rightarrow$	denotes a vector quantity
$-$	denotes an average value over a step length

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## Appendix B

The bulk of this document is concerned with the discussion of the analysis of gas-particle flows which have been shown to be non-isentropic as well as non-isoenergetic in the gas phase. However, one of the original objectives of the RAMP code development was to provide for a single phase (gas) calculation as an option. Computationally this entails simply considering the coefficients reflecting particle contributions to the compatibility equation as numerically equal to zero. As expected, there is no change in entropy and gas total enthalpy along gas streamlines, that is, the flow is seen to be isentropic and isoenergetic. However, for many liquid propellant motor applications a fuel rich mixture of the chamber combustion products is used to film cool the nozzle walls. This results in fuel striations across the nozzle where each streamline has a different energy level associated with it (i.e. non-isoenergetic flow).

This case can be treated with a non-equilibrium chemical analysis by simply specifying the appropriate initial gas total conditions on each streamline. Recall that the pressure-density-velocity form of the compatibility equation is then used. Gas chemical concentrations are required for each streamline so that the gas thermodynamics are computed locally as the solution proceeds downstream in the nozzle-exhaust plume flowfield. However, if the flow is to be considered in chemical equilibrium or equilibrium/frozen, the applicable compatibility relations must reflect the non-isoenergetic nature of the striated flow condition. These relations are developed in the following paragraphs.

To begin, the development of the species continuity equation (starting with equation 2-6) could be replaced by atomic conservation

equations. Moreover if the conservation of those atoms associated with the fuel and the oxidizer were considered then, for steady state,

$$\nabla \cdot (\rho_f \vec{q}) = 0 \quad (B-1)$$

and

$$\nabla \cdot (\rho_o \vec{q}) = 0 \quad (B-2)$$

would result.

If the weight flow ratio of oxidizer to fuel (O/F ratio) is denoted by  $\eta$  then Eq. (B-1) and B-2) are satisfied if,

$$\vec{q} \cdot \nabla \eta = 0 \quad (B-3)$$

and

$$\nabla \cdot (\rho \vec{q}) = 0 \quad (B-4)$$

The assumptions inherent in arriving at Eq. (2-50) did not involve isoenergetic flow so that the momentum equation remains valid. The energy Eq. (2-105), however, remains valid only along each streamline and must be replaced by

$$h + \frac{\vec{q}^2}{2} = H(\eta) \quad (B-5)$$

The compatibility equations which apply along each Mach line are unaltered by the non-isoenergetic analysis since it is constructed based on the momentum equation and the global continuity equation (neither of which are altered by the non-isoenergetic flow phenomena).

The numerical solution to the governing equations is not greatly affected by the modification to the compatibility equation. In Eq. (3-146)

$$\Delta s_{1,2} \Leftarrow -\frac{1}{\bar{T}_{1,2}} \left( \frac{\Delta P_{1,2}}{\bar{\rho}_{1,2}} + \bar{q}_{1,2} \Delta q_{1,2} \right) \quad (B-6)$$

while for a lower pressure boundary

$$(s_3 - s_{1,2}) \Leftarrow \frac{1}{\bar{T}_{1,2}} \left( \frac{\Delta P_{1,2}}{\bar{\rho}_{1,2}} + \bar{q}_{1,2} \Delta q_{1,2} \right) \quad (B-7)$$

The finite difference analog to the streamline equation may be written:

a. interior point

$$\eta_3 = \eta_5$$

b. an upper wall or upper free boundary point

$$\eta_3 = \eta_1$$

c. a lower wall or lower pressure boundary point

$$\eta_3 = \eta_2$$

The oblique shock relations of Section 4 are unaffected since the discussion concerns a single streamline. All references to flow variables are understood to be a function of the O/F ratio. Since that ratio does not change across the shock the only alteration necessary is that of Eq. (4-11) which is altered to read

$$P_2 = P(s_2, q_2, \eta_1), \quad \rho_2 = \rho(s_2, q_2, \eta_1) \quad (B-8)$$

Since the Prandtl-Meyer calculation of Section 5 is also concerned with a single streamline, hence a constant value of  $\eta$ , it to is unaffected.

It can readily be seen from the above discussion that the alterations necessary to incorporate the non-isoenergetic analysis are straightforward from an analytic point of view.

Appendix C

CHEMICAL EQUILIBRIUM CALCULATIONS  
IN GAS-PARTICLE FLOWS



# Appendix C SYMBOLS AND NOTATION

<u>Symbols</u>	<u>Description</u>
A	parameter defined by Eq. (2-46), 1/sec
B <sub>1</sub>	parameter defined by Eq. (2-103)
c <sub>p</sub>	specific heat at constant pressure, ft <sup>2</sup> /sec <sup>2</sup> /°R
H	total enthalpy, ft <sup>2</sup> /sec <sup>2</sup>
M <sub>w</sub>	molecular weight, lbm/lb mole
N	number of moles
N <sub>p</sub>	number of discrete particles
P	pressure, lbf/ft <sup>2</sup>
q	velocity, ft/sec
R	gas constant, ft <sup>2</sup> /sec <sup>2</sup> /°R
S	entropy, ft <sup>2</sup> /sec <sup>2</sup> /°R
T	temperature, °R
u, v	velocity components, ft/sec
V	velocity, ft/sec
$\dot{w}$	flow rate, lbm/sec
x	position coordinate, ft
<u>Greek</u>	
ρ	density, slugs/ft <sup>3</sup>
θ	inclination of the flow vector with respect to the x-axis, deg

## SYMBOLS AND NOTATION (Cont'd)

### Subscripts

c	denotes chamber conditions
g,G	indicates the quantity pertains to the gaseous species
m	indicates the quantity pertains to the gas-particle mixture
p	indicates the quantity pertains to the particle species

### Superscript

j	indicates the quantity pertains to the particle species
---	---

### Appendix C

One of the basic assumptions made in the development of equations describing the gas-particle flow is that there is no mass transfer between the phases. Consequently, calculations treating chemical reactions in gas-particle flows must reflect this assumption. Prozan (Ref. C-1) has shown for chemical equilibrium that the gas thermochemistry calculations can be uncoupled from gas-dynamic considerations. Gas thermodynamic properties (Ref. C-2) for chemical equilibrium or frozen flow are precalculated for nozzle chamber conditions. Tabulated thermodynamic data are then constructed in terms of the independent variables by expanding the flow from the chamber conditions.

Thermochemistry calculations are, however, complicated by condensed species produced in the combustion process. Since the particles are inert, they perform no expansion work. Particle acceleration is accomplished by viscous drag forces exerted by the gas. This, coupled with the gas cooling resulting from expansion, produces dynamic and thermal lags. The work required to accelerate the particles combined with the heat exchanged between the particle and gas phases is a non-isentropic process as shown by the following relations:

$$Tds - \frac{(C_p - R)}{R \rho q \cos \theta} \sum_{j=1}^{NP} \rho^j A^j B_1^j dx = 0; \quad (C-1)$$

$$dH - Tds + \frac{1}{\rho} \sum_{j=1}^{NP} \rho^j A^j \left[ (u - u^j) + \frac{v}{u} (v - v^j) \right] dx = 0. \quad (C-2)$$

The term  $B_1^j$  is a function of the velocity and thermal lags and in essence reflects the loss in work potential. Equation (C-1) indicates that if the algebraic sign on  $B_1^j$  is positive then the change in the entropy level along a streamline will be positive. This is the case when the entropy increase resulting from the work required to accelerate the particle is greater than the heat transferred from the particles to the gas. Consequently there will be a corresponding decrease in gas total enthalpy. It should be noted that just the reverse is true if the increase in work potential due to heat transfer is greater than the work required to accelerate the particles.

The original chemical equilibrium formulation (Ref. C-2) assumed that the gas and particles comprising the flow mixture were in dynamic and thermal equilibrium. Condensed species resulting from the combustion process were treated in the same fashion as the gas chemical species in that chemical reactions were permitted. The expansion process was then isentropic since gas-particle equilibrium was assumed. However, the correct thermochemical properties can be obtained if certain modifications to the analysis are made. A step-by-step description of the modifications is contained in the following paragraphs.

First the adiabatic flame calculation is made for the gas-particle mixture (Step 1, Fig. C-1). The mixture concentrations comprise the gas chemical species as well as the particulate species. Examination of the concentrations permits identification of the particulates. Since the assumption has been made that the gas and particulate phases do not exchange mass, the chemical reactions between the phases are suppressed by removing these from the list of possible reactions. Particle energy and mass are removed from the mixture (Step 2). The particle loading is then calculated from the relation

$$\frac{\dot{w}_p}{\dot{w}_g} = \frac{M_{wp} N_p}{M_{wg} N_g} \quad (C-3)$$

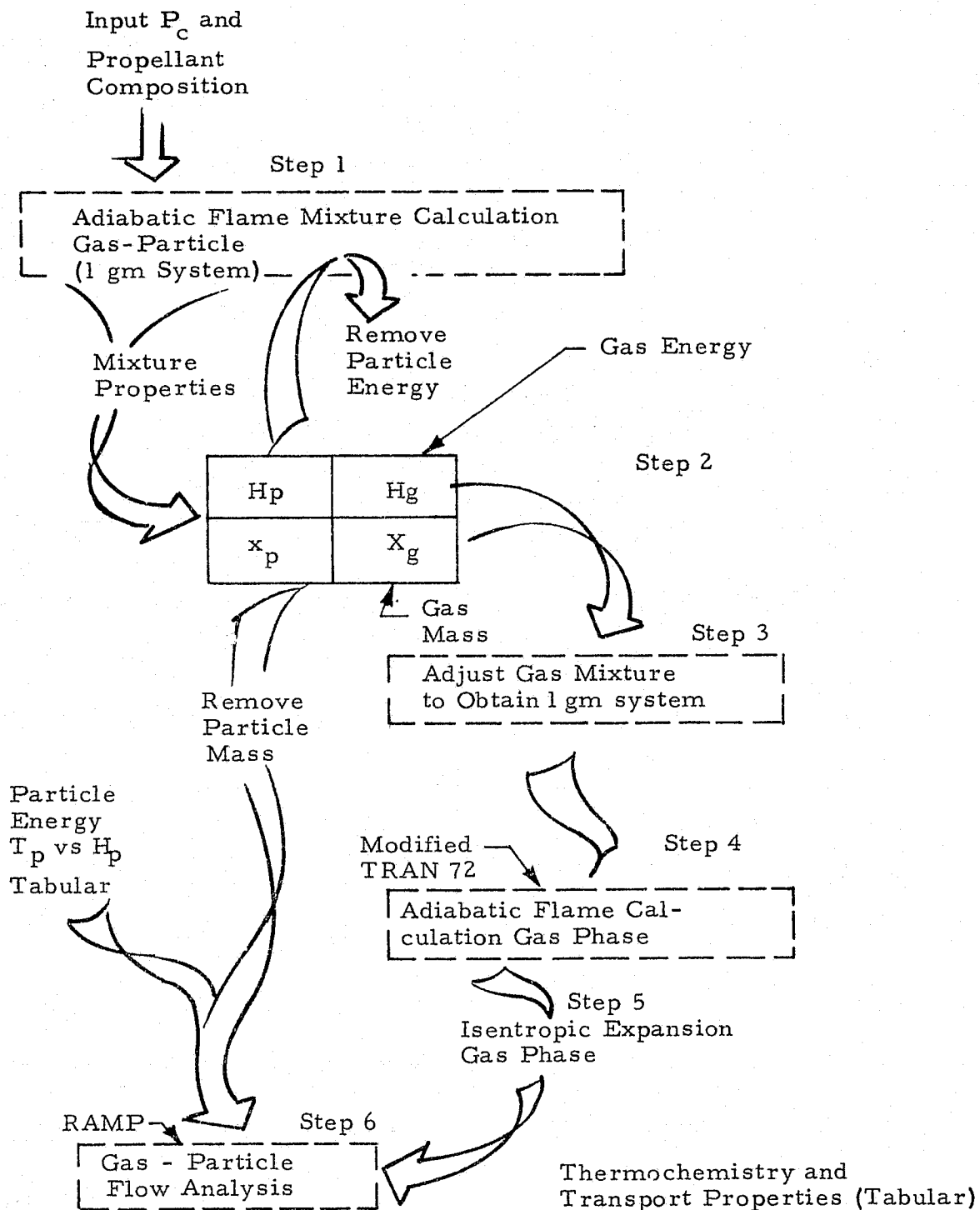


Fig.C-1 - Schematic of Two-Phase Thermochemical Calculations

Equation (C-3) gives the total particle mass loading relative to the gas. Individual particle loadings are obtained from empirical data for a given motor class and size. These data along with the particle enthalpy - temperature data (Ref. C-3) which defines the energy removed from the mixture become input data to the RAMP code. Next, the remaining gas energy and molar concentrations are then readjusted to obtain a one gram system (Step 3). The adiabatic flame calculation is made for the gas phase at the chamber pressure and temperature computed for the gas-particle mixture (Step 1).

Equations (C-1) and (C-2) indicated the gas phase expansion process to be non-isentropic. Hence the thermochemistry calculation must reflect this condition. This is accomplished in the following manner.

Equation (C-2) defines the change in gas total enthalpy as a function of entropy change and particle velocity lags. A schedule of gas total enthalpy changes can now be defined by the relation

$$H_{G_i} = H_c + \Delta H_i ; i = 1, n \quad (C-4)$$

The values of  $\Delta H_i$  are defined to cover the expected change in gas total enthalpy for a given analysis.

Figure C-2 shows a typical schematic of an H-S diagram for gas-particle flows. Each point  $H_{G_i}, P_c$  defines a particular point on the diagram from which the flow can be isentropically expanding for the corresponding value of  $S_1$ . However, the flow is non-isentropic so that for each  $H_{G_i}, P_c$  an expansion from one or more states defined by a change in entropy must be made. This is depicted schematically in Fig. C-2. Computationally this is accomplished by inputting to the code a schedule of  $\Delta H_i$  and a corresponding change in entropy for each  $\Delta H_i$ . The chamber calculation is made for the gas phase (Step 3,

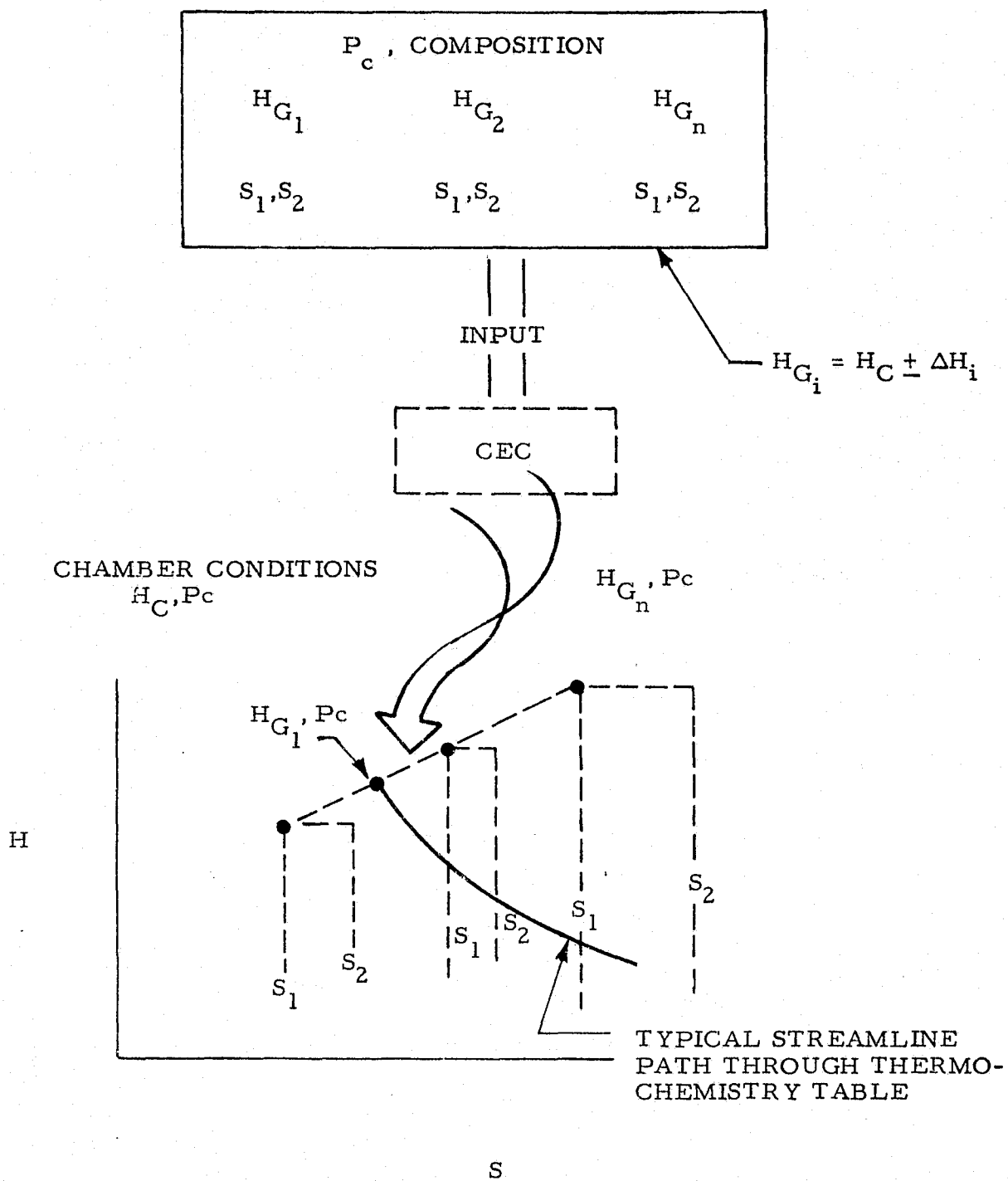


Fig. C-2 - Schematic of Thermochemistry Table Construction

Fig. C-1) and an isentropic expansion from the reference conditions calculated for the number of entropy levels desired. These data are generated in tabular form as a function of  $\Delta H_i$ ,  $\Delta S_j$  and  $V_k$  respectively as indicated by the following schematic. This process is subsequently

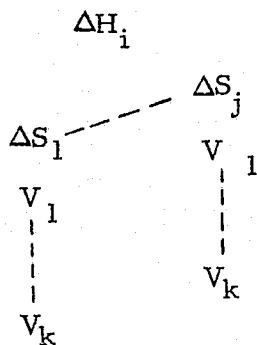


Fig. C-3 - Schematic of Tabulated Thermochemistry Data

repeated for each  $\Delta H_i$ . Since for a chemical equilibrium analysis the independent variables chosen are  $\Delta H$ ,  $\Delta S$  and  $V$  the local thermodynamic and transport properties at a given point in the flow are obtained by interpolating within the tables.



Appendix C  
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Appendix D  
NON-CONTINUUM FLOW EXPANSIONS

# Appendix D SYMBOLS AND NOTATION

<u>Symbol</u>	<u>Description</u>
Kn	Knudsen number, dimensionless
M	Mach number, dimensionless
N	number of collisions per molecule required for the gas to achieve relaxation
R	gas constant, $\text{ft}^2/\text{sec}^2/^\circ\text{R}$
Re	Reynolds number, dimensionless
Ro	characteristic dimension of the flow field, ft
S	distance along a streamline, ft
T	temperature, $^\circ\text{R}$
V	gas velocity, ft/sec
$\bar{V}$	mean molecular velocity, ft/sec
<u>Greek</u>	
$\gamma$	specific heat ratio, dimensionless
$\mu$	viscosity, $\text{lbf}\cdot\text{sec}/\text{ft}^2$
$\rho$	density, $\text{slug}/\text{ft}^3$
$\lambda$	mean free path, ft

## Appendix D

### D.1 DETERMINATION OF THE LOCAL KNUDSEN NUMBER, Kn

The expansion of exhaust gases to very low back pressures results in a decrease in the gas pressure and density. This in turn produces a decrease in the number of intermolecular collisions. At some point, the effect of intermolecular collisions will cease to be a governing factor in determining the flowfield characteristics. The local Knudsen number (Kn) is used to indicate the local flow regime, i.e., continuum, vibrationally frozen, rotationally frozen or translationally frozen. The local Knudsen number is defined as the ratio of the mean free path (which is a measure of the average distance between intermolecular collisions) to a characteristic dimension of the flowfield. Kn is given by this relation (Ref. D-1):

$$Kn = \left( \frac{V}{\bar{V}} \lambda \right) \left( \frac{1}{T} \frac{dT}{dS} \right) = \frac{1}{N}$$

In the relation for Kn, the mean free path is represented by  $\frac{V}{\bar{V}} \lambda$  and the characteristic dimension is represented by  $T/\frac{dT}{dS}$ . The method employed here is to utilize the "sudden freeze" approximation (Ref. D-1) to determine when an energy mode freezes such that the gas temperature cannot relax fast enough to maintain the equilibrium rate of change. The RAMP program computes a local Kn at each mesh point and compares it to  $1/N_i$  (the inverse of the collision number) to determine if the  $i^{th}$  energy mode has frozen.

Assuming that the mean molecular speed  $\bar{V}$  is determined by the Maxwell distribution function

$$\bar{V} = \left( \frac{8}{\pi} RT \right)^{1/2} \quad (\text{Ref. D-2})$$

where  $R$  is the particular gas constant. Introducing the local Mach number  $M$ ,  $Kn$  can be written:

$$Kn = \left( \frac{\pi}{8} \gamma \right)^{1/2} = M \lambda \left[ \frac{d(\ln T)}{dS} \right], \quad d(\ln T) = \frac{dT}{T}.$$

For rarefied gases the mean free path can be related to the viscosity  $\mu$

$$\lambda = \frac{\mu}{0.499 \rho \bar{V}} \quad (\text{Ref. D-3})$$

where  $\rho$  is the local gas density. Let  $\bar{S} = \frac{S}{R_o}$ , where  $R_o$  is the characteristic dimension of the flowfield. Then  $dS = R_o d\bar{S}$ . With the above relation  $\lambda = f(\mu)$ , and introducing the characteristic dimension  $R_o$ ,  $Kn$  can be written:

$$Kn = \frac{\pi \gamma M^2}{8(.499)} \left( \frac{\mu}{\rho \bar{V} R_o} \right) \frac{d(\ln T)}{d\bar{S}}.$$

But the local Reynolds number  $Re$  is:

$$Re = \frac{\rho \bar{V} R_o}{\mu}$$

Therefore,

$$Kn = 0.25050 \pi \frac{\gamma M^2}{Re} \frac{d(\ln T)}{d\bar{S}}.$$

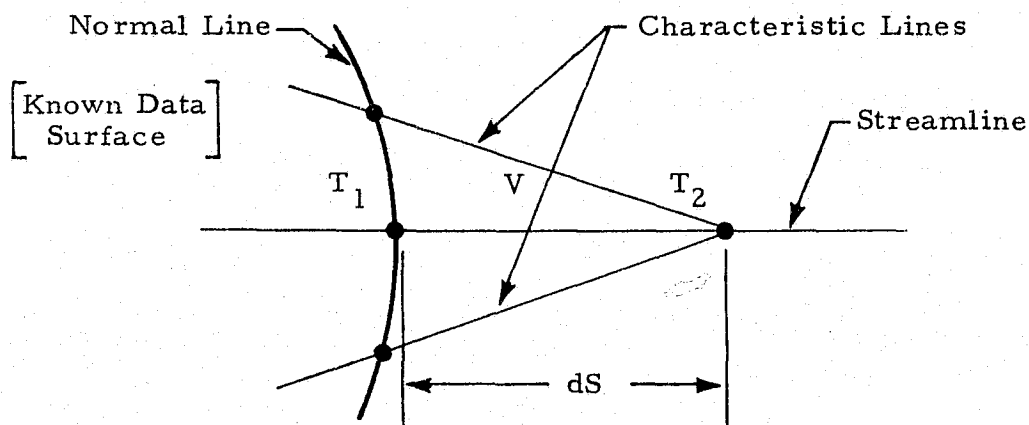
As a first approximation to the derivative  $\frac{d(\ln T)}{d\bar{S}}$ , let

$$\frac{d(\ln T)}{d\bar{S}} \simeq \frac{\ln (T_1/T_2)}{dS/R_0}$$

so that the local Knudsen number, Kn as coded is given by:

$$Kn = 0.25050 \pi \frac{\gamma M^2}{Re} \left[ \frac{\ln T_1 - \ln T_2}{dS/R_0} \right]$$

The characteristic mesh is shown below to illustrate the flowfield variables used to determine the local Knudsen number



## D.2 NON-CONTINUUM FLOW REGIME CRITERIA

The criteria used by the RAMP program to determine the "freezing" of particular energy modes is to compare the local Knudsen number (Kn) with the reciprocal of the collision number. When Kn is equal to or greater than the reciprocal of the collision number for the vibrational, rotational, or translational modes that particular mode is considered to be frozen. The freezing of the first two energy modes (vibrational and rotational) represents a transitional computational region from the continuum to free molecular regimes. During this transitional period the

gaseous flowfield calculation is performed using the characteristic equations with the thermodynamic properties modified to reflect the respective flow regime. That is, the gaseous ratio of specific heats is altered to reflect the transition to a non-continuum regime. Once the translational energy mode has frozen the flowfield becomes "free molecular." In this case the solution becomes a source flow solution with a constant gaseous ratio of specific heats.

#### D.2.1 Non-Continuum Flowfield Gas Properties

The energy modes of the gaseous flowfield freeze one at a time in order of increasing energy requirements. The vibrational mode freezes first, the ratio of specific heats ( $\gamma$ ) is set to  $7/5$  and the flowfield solution proceeds in the conventional streamline normal fashion. See Ref. D-4 for the characteristics equations. The rotational mode freezes next in which case  $\gamma$  is computed as a function of  $Kn$  (third order polynomial) and varies from  $7/5$  to  $5/3$ . This flowfield solution is not altered except for the determination of  $\gamma$ . The last energy mode to freeze is the translational mode. Once the translational mode has frozen, the flowfield is considered to be in the free molecular regime.  $\gamma$  is set to  $5/3$  and is held constant. The effect of the freezing of the three gaseous internal energy modes on the gas ratio of specific heats is depicted pictorially in Fig. D-1.

Once the flowfield calculation has switched to the free molecular regime the gas streamlines remain straight and the gas velocity which is directed along streamlines remains constant. The gas constant, temperature, and flow angle also remain constant. The gas density varies inversely as the streamtube area and the gas pressure is obtained from the equation of state.

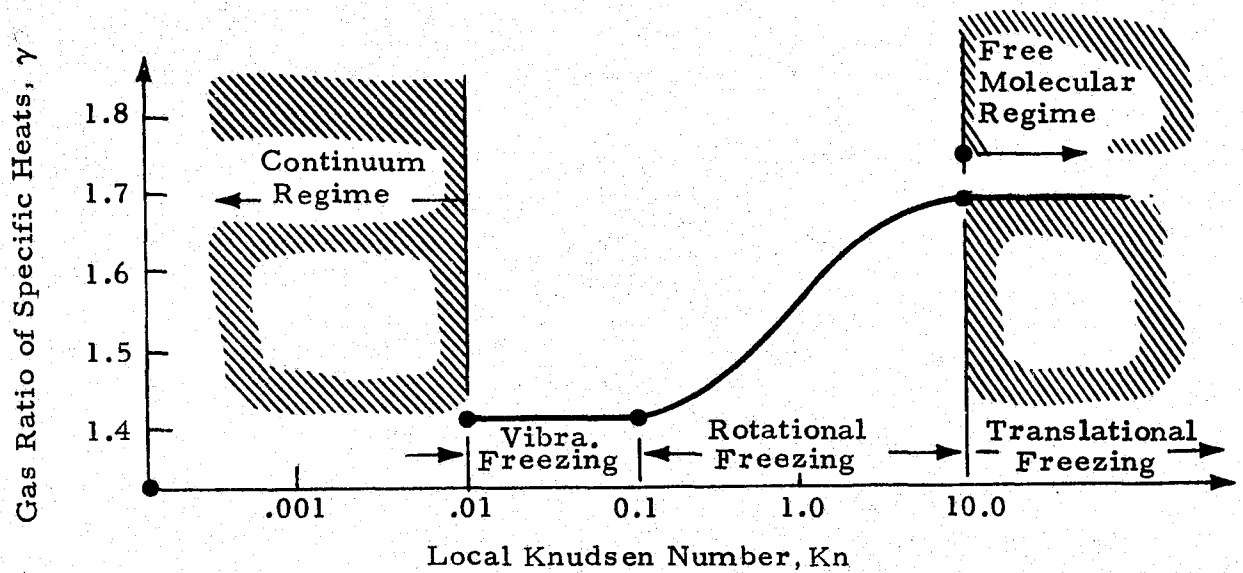


Fig. D-1 - Variation of Gamma with the Local Knudsen Number

#### D.2.2 Non-Continuum Flowfield Particle Properties

If particles are present in the free molecular flowfield their status is computationally very similar to the gas. Particle velocity, temperature, and enthalpy remain constant. The particle flow angle remains constant, and the particles are directed along straight streamlines. The particle and gas streamlines are not necessarily parallel. The particle density varies inversely as particle stream tube cross-sectional area.



Appendix D  
REFERENCES

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**Appendix E**  
**INTEGRATION OF THE FINITE RATE CHEMICAL**  
**KINETIC EQUATIONS**

## Appendix E

In flow problems where the gas may be considered in equilibrium (chemical and thermodynamic) at every point, two parameters are sufficient to define any of the other thermodynamic variables, either by assuming a perfect gas or by utilizing the results of chemical equilibrium calculations of the gases involved. If the gases cannot be considered to be in equilibrium, a point-by-point evaluation of its composition via integrating the individual species continuity equations is required. This appendix addresses the procedure utilized in the RAMP program to perform such calculations.

A detailed description of the rate processes that occur in rocket exhaust flows requires that a myriad of mechanisms be considered to include all the possible chemical reactions of dissociation, formation, recombination, etc.

All of these, however, can be treated with a very general formalism. In the form usually quoted in chemical kinetics (Ref. E-1) the phenomenological law of mass action states that the rate of a reaction is proportional to the product of the concentrations of the reactants. Thus, for a general reaction of the form



the net rate of production  $\dot{w}_i$  for any participating species for which the stoichiometric coefficients  $\nu_i'$  and  $\nu_i''$  are not equal can then be written as

$$\dot{w}_i = k_f \prod_{i=1}^N (C_i)^{\nu_i'} - k_b \prod_{i=1}^N (C_i)^{\nu_i''} \quad (E.2)$$

where  $C_i$  is the species concentration defined as  $C_i = \rho F_i$  with  $\rho$  being the density and  $F_i$  being the species mol/mass ratio, respectively. Assuming small deviations from equilibrium, the forward and backward reaction rate constants,  $k_f$  and  $k_b$ , respectively, can be related to the concentration equilibrium constant and to the pressure equilibrium constant as follows:

$$\frac{k_f}{k_b} = K_c = K_p (\bar{R} T)^{\sum_{i=1}^N (\nu_i' - \nu_i'')} \quad (E.3)$$

The significance of the pressure equilibrium constant  $K_p$  is that it can be easily evaluated for any reaction using tabulated values of  $K_f$  the equilibrium constant for formation from the elements. Values of  $K_f$  are commonly tabulated in conjunction with specific heats, entropies and enthalpies as a function of temperature, and are available in general for most species. An equally convenient method exists for determining  $K_p$  from the change of free energy accompanying the reaction, i.e.,

$$K_p = \exp(-\Delta G / \bar{R} T) \quad (E.4)$$

where  $\Delta G$  is the change in free energy during the reaction process. Free energy values are also available for most species in tabular form. This method is used to compute  $K_p$  in the RAMP program.

Using Eq. (E.3) to eliminate the backward rate constant from Eq. (E.2) the general production rate equation can be written as

$$\dot{w} = k_f \left\{ \prod_{i=1}^N (C_i)^{\nu_i'} - \frac{\sum_{i=1}^N (\nu_i'' - \nu_i')}{K_p} \prod_{i=1}^N (C_i)^{\nu_i''} \right\} \quad (E.5)$$

Finally, the net rate of production for any species participating in the reaction, either as a reactant or as a reaction product, is then given by

$$\dot{w}_i = (\nu_i'' - \nu_i') \dot{w} \quad (E.6)$$

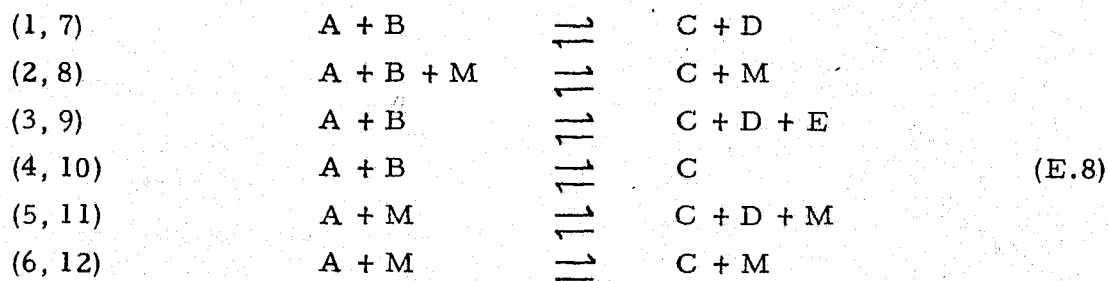
Since most reaction systems involve numerous simultaneous reactions, the net rate of production of species  $i$  usually equals a sum of terms. Thus, for an arbitrary number of  $M$  simultaneous reactions, all involving species  $i$ ,

$$\dot{w}_i = \sum_{k=1}^M \dot{w}_{i,k} \quad k = 1, \dots, M \quad (E.7)$$

where  $\dot{w}_{i,k}$  denotes the net rate of production of species  $i$  due to reaction  $k$  only.

For reasons of computational speed and efficiency, the program contains explicit expressions, as obtained from Eq. (E.5), for the most commonly encountered reaction mechanisms. Twelve types of reaction mechanisms are considered as possible contributors to the calculation of the net rate of production,  $\dot{w}_i$ :

Reaction type



Reaction types (7) through (12) correspond to reaction types (1) through (6), but proceed in the forward direction only.

To reduce roundoff and truncation errors, the forward and backward rates for each reaction are computed separately. All contributions to the

molar rate of production of a given species are then computed and added algebraically to form matrix coefficients (discussed later). Since reaction types (7) through (14) proceed in the forward direction only, the second term on the right-hand side of Eq. (E.5) is disregarded in calculating the contributions to the coefficient matrix.

In reactions (2), (5) and (6) as well as in (8), (11) and (12), M denotes a third body species which can be specified. For these reactions the situation often occurs where for various third bodies the respective rate constants differ only by a constant multiplier. These multipliers can be considered as third body efficiencies or weighting factors. If such a case is encountered, the third body species mole mass ratio  $F_M$  becomes effectively a fictitious mole mass ratio, consisting of the weighted sum over all those species having a nonzero weighting factor, i.e.,

$$F_M = \sum_i f_i F_{M_i} \quad (E.9)$$

where  $f_i$  are the weighting factors.

The forward rate constant,  $k_f$ , is generally a function of temperature. It is most often expressed in Arrhenius form. Again, for speed and efficiency in computation, the rate constants are divided into five types:

#### Rate Constant Type

(1)	$k_f = A$	
(2)	$k_f = AT^{-N}$	
(3)	$k_f = A \exp(B/RT)$	(E.10)
(4)	$k_f = AT^{-N} \exp(B/RT)$	
(5)	$k_f = AT^{-N} \exp(B/RT^M)$	

The equations presented in this appendix provide a very general formalism for the evaluation of various rate processes. The specification of particular systems and associated rate constants is up to the program user.

Considering now the general species continuity equation

$$\rho \vec{q} \cdot \nabla C_i = \dot{w} \quad (E.11)$$

and making use of the foregoing discussion of the rate process we now proceed to describe a calculational technique for determining the individual species composition on a point-by-point basis. The description of this process is substantially simplified if Eq. (E.5) is specialized to a particular reaction type, say number (7) from Eq. (E.8) which is a one-way, two-body reaction.



the net production rate for this process is

$$\dot{w} = k_f \rho^2 F_A F_B \quad (E.13)$$

and the species continuity equation for species B then becomes

$$\rho \vec{q} \cdot \nabla F_B = k_f \rho^2 F_A F_B \quad (E.14)$$

which along streamlines becomes

$$\rho q \frac{\partial F_B}{\partial S} = k_f \rho^2 F_A F_B \quad (E.15)$$

This equation can readily be solved using finite difference techniques employing explicit relationships, such as Euler or more sophisticated schemes, such as Runge-Kutta. The step size for integrating this equation however is severely limited by stability criteria. It can be seen from Eq. (E.15) that the rate of change of a species along the streamline becomes increasingly larger as the flow speed is slowed, the density increased, or for fast reaction rates. In rocket engine problems, combinations of slow speeds, high densities and fast reaction rates (i.e., quasi-equilibrium) are quite common and integration step sizes so small (i.e.,  $< 10^{-8}$  meters) are encountered that the solution becomes impractical in terms of computation time.

For this reason, the technique described in Ref. E-2 based on a linearization of the production rates was utilized. Writing Eq. (E.15) in finite difference form over a streamline step from station  $n$  to  $n+1$ .

$$F_{B_{n+1}} = F_{B_n} - \frac{\Delta S \rho}{q} F_{A_{n+1}} F_{B_{n+1}} \quad (E.16)$$

And evaluating all the species concentrations at the downstream point results in a set of simultaneous nonlinear algebraic equations. In order to solve these equations we must then linearize the term  $F_{A_{n+1}} F_{B_{n+1}}$  which is accomplished following the lead of Ref. E-2. If this term is expanded in terms of its values at station  $n$  along with the increments over  $n$  to  $n+1$  we can obtain the following expression.

$$F_{A_{n+1}} F_{B_{n+1}} = F_{A_n} F_{B_{n+1}} + F_{B_n} F_{A_{n+1}} \quad (E.17)$$

neglecting products of differentials which are assumed to be of second order importance. Equation (E.16) can now be written in its linearized form.

Let  $C = \Delta S \rho / q$

$$\begin{aligned} F_{B_{n+1}} &= F_{B_n} - C \left[ F_{A_n} F_{B_{n+1}} + F_{B_n} F_{A_{n+1}} \right] \\ \text{and} \quad F_{A_{n+1}} &= F_{A_n} - C \left[ F_{A_{n+1}} F_{B_n} + F_{B_{n+1}} F_{A_n} \right] \end{aligned} \quad (E.18)$$

Equation (E.18) can then be expressed in terms of a set of unknowns and calculable coefficients,  $C$ . Rewriting these we obtain

$$\begin{aligned} F_{A_{n+1}} &= F_{A_n} - C F_{A_n} (F_{B_{n+1}}) - C F_{B_n} (F_{A_{n+1}}) \\ F_{B_{n+1}} &= F_{B_n} - C F_{B_n} (F_{A_{n+1}}) - C F_{A_n} (F_{B_{n+1}}) \end{aligned} \quad (E.19)$$



$$\begin{aligned}
 F_{A_{n+1}} (1 + C F_{B_n}) + (C F_{A_n}) F_{B_{n+1}} &= F_{A_n} \\
 F_{A_{n+1}} (C F_{B_n}) + (1 - C F_{A_n}) F_{B_{n+1}} &= F_{B_n}
 \end{aligned}
 \tag{E.19}$$

Cont'd

A matrix can now be formed using totally known information.

$$\begin{bmatrix} 1 + C F_{B_n} & C F_{A_n} \\ C F_{B_n} & 1 - C F_{A_n} \end{bmatrix} \begin{bmatrix} F_{A_{n+1}} \\ F_{B_{n+1}} \end{bmatrix} = \begin{bmatrix} F_{A_n} \\ F_{B_n} \end{bmatrix}
 \tag{E.20}$$

The matrix  $[A] [X] = [B]$  is then solved for the unknown compositions  $F_{A_{n+1}}$ ,  $F_{B_{n+1}}$  via a triangulation technique. Although consuming more time per integration step than an explicit formulation, the implicit technique employed here is unconditionally stable permitting much larger step sizes, thus allowing solutions to be obtained for problems where the small steps required by the explicit technique prevented even the consideration of the case. Finally it should be recalled that an extremely simple case was chosen only for purposes of illustration and the general technique coded in RAMP handles many species (see input guide) with multiple reactions.

#### REFERENCES

- E-1. Williams, F.A., Combustion Theory, Addison-Wesley Publishing Co., Inc., Reading, Mass., 1965.
- E-2. Moretti, G., "A New Technique for the Numerical Analysis of Non-Equilibrium Flows," AIAA J., Vol. 3, No. 2, February 1965.